Problem Set III

Macroeconomics II

Armando Näf*

August 26, 2020

1 No-Ponzi-game condition

Consider an infinitely lived household with assets $a_0 < 0$ at time $t = 0$. The household compares different options of servicing the debt, as specified below. For each option, check whether the household satisfies the condition $\lim_{T\to\infty} a_{T+1} \geq 0$ and/or the no-Ponzi-game condition $\lim_{T\to\infty} q_T a_{T+1} \geq 0$,, where $q_T = \frac{1}{R_1 R_2 \cdots R_T}$ and compute the present discounted value of the debt service (discounting at the gross interest rate R). Assume that income is sufficiently high such that each option is feasible.

1. In period 0, fully repay all debt (i.e., pay $-Ra_0$).
2. In each period $t \geq 0$, pay $-xa_t$, where $0 < x < R$.
3. In each period $t \geq 0$, don't pay anything.

2 Intertemporal budget constraint with infinite horizon

Show that the no-Ponzi-game condition together with the dynamic budget constraints implies the intertemporal budget constraint

$$a_0 R_0 + \sum_{t=0}^{\infty} q_t (w_t - c_t) \geq 0.$$

3 Non-Geometric Discounting and Time-Consistency

Consider the following three period model, $t = 0, 1, 2$. At date $t = 0$ preferences are given by $u(c_0) + \beta(u(c_1) + u(c_2))$, but at time $t = 1$ preferences are given by $u(c_1) + \beta u(c_2)$. For simplicity, assume that $w_t = w$ and $R_t = 1, t = 0, 1, 2$.

Set up the problem of a household under commitment at $t = 0$ and the problem of a household that re-optimizes at $t = 1$. Derive the Euler equations of the two problems and show that the household that re-optimizes every period would choose a different consumption plan.

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

4 Dynamic programming

Let $V_t(a_t)$ denote the maximal utility which the household can achieve from period t on if it enters the period with assets a_t.

1. Show that

$$\begin{aligned} V_t(a_t) = \max_{\{c_s, a_{s+1}\}_{s=t}^{T}} & \sum_{s=t}^{T} \beta^{s-t} u(c_s) \\ \text{s.t. } & a_{s+1} = a_s R_s + w_s - c_s \text{ for } s = t, t+1, \dots, T \\ & a_t \text{ given} \\ & a_{T+1} \geq 0 \end{aligned}$$

can be rewritten as

$$\begin{aligned} V_t(a_t) = \max_{c_t, a_{t+1}} & \left[u(c_t) + \beta V_{t+1}(a_{t+1}) \right] \\ \text{s.t. } & a_{t+1} = a_t R_t + w_t - c_t \\ & a_t \text{ given.} \end{aligned}$$

Now consider the Bellman equation

$$V_t(a_t) = \max_{a_{t+1}} \left[u(a_t R_t + w_t - a_{t+1}) + \beta V_{t+1}(a_{t+1}) \right].$$

Note that $V_{T+1}(a_{T+1}) = 0$, implying the optimal choice $a^*_{T+1} = 0$.

2. By backward induction, solve for the value functions $V_T(a_T)$ and $V_{T-1}(a_{T-1})$, and for the policy functions $a_T(a_{T-1})$ and $a_{T-1}(a_{T-2})$, given the assumptions that $u(c_t) = \ln(c_t)$ and $w_t = 0$ for all t.

3. Using the Bellman equation and the envelope theorem, derive the Euler equation.

5 Dynamic programming with infinite horizon: guess and verify

Assume that $u(c_t) = \ln(c_t)$, $w_t = 0$ and $R_t = R$ for all t. Thus the Bellman equation is given by

$$V(a) = \max_{a_+} \ln(aR - a_+) + \beta V(a_+).$$

Solve for the value function $V(a)$ and for the policy function $g(a)$ in the infinite horizon case using a guess and verify approach. An educated guess for the two functions is

$$\begin{aligned} V(a) &= F \ln a + G \\ g(a) &= HaR, \end{aligned}$$

where F, G and H are unknown coefficients. Proceed as follows:

1. Using the Bellman equation, show that given the guess for $V(a)$, the policy function is indeed of the form $g(a) = HaR$. Solve for H as a function of β, F and G.

2. Using the Bellman and your result from a., use the method of undetermined coefficients to solve for the coefficients F, G and H.