Problem Set VII

Macroeconomics II

Solutions

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1 Solution: Portfolio choice and C-CAPM

Consider an environment with risk-free labor income and two assets: asset 1 is risk-free while asset 2 is risky with a positive excess return. Show that an investor chooses a strictly positive exposure to the risky asset.

Hint: Use the Euler Equations and proceed with a proof by contradiction.

Solution:

We have two assets, a risky asset with return R_1^r and a risk-free asset with return R_1^f. The risky asset has a positive excess return and therefore

$$\mathbb{E}_t\left[R_1^r - R_1^f\right] > 0$$

From the Euler equations of the consumer we know that following two conditions must hold

$$u'(c_t) = \beta\,\mathbb{E}_t\left[R_1^r u'(c_{t+1})\right]$$
$$u'(c_t) = \beta\,\mathbb{E}_t\left[R_1^f u'(c_{t+1})\right]$$

Suppose that the consumer would choose not to buy any of the risky asset. Then by definition the outcome at $t+1$ would have no uncertainty and consumption at $t+1$ would be known. The Euler equations can then be written as

$$u'(c_t) = \beta\,\mathbb{E}_t\left[R_1^r\right] u'(c_{t+1})$$
$$u'(c_t) = \beta R_1^f u'(c_{t+1})$$

However combining the two equations yields $\mathbb{E}_t\left[R_1^r\right] = R_1^f$ or $\mathbb{E}_t\left[R_1^r - R_1^f\right] = 0$. This is a contradiction, since we know that the risky asset has a positive excess return.

It follows that the household will always hold at least a marginal unit of the risky asset, because the marginal utility of holding $\epsilon > 0$ units of the risky asset is larger than holding an additional small unit of the risk-free asset (where ϵ is a small positive number.

The same conclusion can be found by using the C-CAPM model. Let M_t denote the stochastic discount factor of the household where $M_t = \beta\frac{u'(c_t)}{u'(c_{t-1})}$. Then from the Euler equation we know that

$$1 = \mathbb{E}_t\left[M_{t+1}R_{t+1}^i\right]$$

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

for all available assets i. In particular this must also be true for the risk-free asset, $1 = \mathbb{E}_t\left[M_{t+1}R^f_{t+1}\right]$ or $\mathbb{E}_t\left[M_{t+1}\right] = \frac{1}{R^f_{t+1}}$. It follows that

$$\mathbb{E}_t\left[M_{t+1}\left(R^j_{t+1} - R^f_{t+1}\right)\right] = 0$$

for all risky assets j. Using the definition of covariance this yields

$$\begin{aligned}
\mathbb{E}_t\left[M_{t+1}\right]\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right] + \mathbb{C}\text{ov}_t\left[M_{t+1}, \left(R^j_{t+1} - R^f_{t+1}\right)\right] &= 0 \\
\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right] &= -\,\mathbb{C}\text{ov}_t\left[M_{t+1}, \left(R^j_{t+1} - R^f_{t+1}\right)\right]\frac{1}{\mathbb{E}_t\left[M_{t+1}\right]} \\
\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right] &= -\,\mathbb{C}\text{ov}_t\left[M_{t+1}, R^j_{t+1}\right]R^f_{t+1}
\end{aligned}$$

Where the covariance is zero if the consumer holds no risky assets, because M_{t+1} is constant. Therefore, we derive the same conclusion and reach a contradiction to the assumption that the household holds no risky assets. Intuitively an asset must have a high excess return if it has a strong negative covariance with the stochastic discount factor. In other words, when the stochastic discount factor is low meaning that the expected consumption tomorrow (c_{t+1}) is larger than todays consumption (c_t), then the households wants to be compensated by a high excess return. This implies that households want a high excess return if they buy an asset, that yields a higher return in a state, in which they are already well off.

2 Solution: Asset pricing and bubbles

1. *Use the definition of a return $R_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$ to rewrite the Euler equation $u'(c_t) = \beta\mathbb{E}_t[u'(c_{t+1})R_{t+1}]$ as an asset pricing equation.*

   Solution:

$$\begin{aligned}
u'(c_t) &= \beta\,\mathbb{E}_t\left[u'(c_{t+1})R_{t+1}\right] \\
u'(c_t) &= \beta\,\mathbb{E}_t\left[u'(c_{t+1})\frac{p_{t+1}+d_{t+1}}{p_t}\right] \\
p_t &= \mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}(p_{t+1}+d_{t+1})\right] \\
p_t &= \mathbb{E}_t\left[M_{t+1}(p_{t+1}+d_{t+1})\right] \qquad (1)
\end{aligned}$$

   where $M_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$ is the stochastic discount factor of the household.

Consider the special case of linear utility and constant dividend payments d per period in a deterministic, infinite horizon environment.

2. *Assuming the absence of bubbles, determine the asset price p_t.*

   Solution:

   For a household with linear utility, the stochastic discount factor M_{t+1} is constant and given by β.

Equation (1) then becomes

$$
\begin{aligned}
p_t &= \mathbb{E}_t\left[M_{t+1}(p_{t+1}+d_{t+1})\right] \\
p_t &= \mathbb{E}_t\left[\beta(p_{t+1}+d)\right] \\
p_t &= \mathbb{E}_t\left[\beta(\mathbb{E}_{t+1}\left[\beta(p_{t+1}+d)\right]+d)\right] \\
&\vdots \\
p_t &= \underbrace{\mathbb{E}_t\left[\sum_{s=1}^{\infty}\beta^s d\right]}_{\text{fundamental value}} + \underbrace{\lim_{s\to\infty}\mathbb{E}_t\left[\beta^s p_{t+s}\right]}_{\text{bubble term}} \\
p_t &= \frac{\beta d}{1-\beta}
\end{aligned}
$$

Where we reach the last line by ruling out asset bubbles, which implies that the bubble term is equal to zero. Further, we know that the fundamental values have no uncertainty due to the combination of linear utility and constant divident flows.

3. *Show that $p_t = \left(\frac{1}{\beta}\right)^t b + \frac{\beta d}{1-\beta}$ is a solution to the asset pricing equation. Why is the term $\left(\frac{1}{\beta}\right)^t b$ called bubble component? What happens as t increases?*

Solution:

Guess and verify. Use the given guess and plug it into the asset pricing equation from (1) with linear utility and constant dividend flows

$$
\begin{aligned}
p_t &= \mathbb{E}_t\left[M_{t+1}(p_{t+1}+d_{t+1})\right] \\
p_t &= \mathbb{E}_t\left[\beta(p_{t+1}+d)\right] \\
\left(\frac{1}{\beta}\right)^t b + \frac{\beta d}{1-\beta} &= \mathbb{E}_t\left[\beta\left(\left(\frac{1}{\beta}\right)^{t+1} b + \frac{\beta d}{1-\beta} + d\right)\right] \\
&= \beta\left(\frac{1}{\beta}\right)^{t+1} b + \beta\frac{d}{1-\beta} \\
p_t &= \left(\frac{1}{\beta}\right)^t b + \frac{\beta d}{1-\beta}
\end{aligned}
$$

That's it, we showed that the guess is correct. The asset price today p_t is given by the fundamental value of the asset, i.e. the sum of discounted dividends given by $\frac{\beta d}{1-\beta}$ and an additional bubble term $\left(\frac{1}{\beta}\right)^t b$. The bubble term is unrelated to fundamental values and can only be explained by an expected increase of the price in the next period. For the asset to have an increase in the price in the next period, that is unrelated to fundamentals, there must be a further increase in the price in the period after and so on. The bubble therefore grows at rate $\frac{1}{\beta} > 1$. In the limit this bubble term converges to infinity as $\lim_{t\to\infty}\left(\frac{1}{\beta}\right)^t \to \infty$.

3 Solution: Derivation of the asset pricing kernel

Consider an environment with two periods and two states of nature in the second period, h and l, both of which are equally likely. There are two assets. Asset 1 pays (1, 1) and costs 1. Asset 2 pays (2, 0) and costs 1 as well. Derive the asset pricing kernel and characterize equilibrium consumption.

Solution:

Using the definition of a return $R_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$, we can rewrite the Euler equation as an asset pricing equation:

$$p_0 = \mathbb{E}_0\left[\frac{\beta u'(c_1)}{u'(c_0)}(p_1 + d_1)\right]$$

Such an equation holds for each asset the household chooses to hold. We plug in the information about the two assets and realise that $p_1 = 0$ for both assets.

- Asset 1:

$$\begin{aligned} p_0^1 &= \mathbb{E}_0\left[\frac{\beta u'(c_1)}{u'(c_0)}(p_1^1 + d_1^1)\right] \\ p_0^1 &= \left(\pi_h \frac{\beta u'(c_{1,h})}{u'(c_0)}(p_1^1 + d_1^1(h)) + \pi_l \frac{\beta u'(c_{1,l})}{u'(c_0)}(p_1^1 + d_1^1(l))\right) \\ 1 &= \left(\frac{1}{2}\frac{\beta u'(c_{1,h})}{u'(c_0)}(1) + \frac{1}{2}\frac{\beta u'(c_{1,l})}{u'(c_0)}(1)\right) \qquad (2) \end{aligned}$$

- Asset 2:

$$\begin{aligned} p_0^2 &= \mathbb{E}_0\left[\frac{\beta u'(c_1)}{u'(c_0)}(p_1^2 + d_1^2)\right] \\ p_0^2 &= \left(\pi_h \frac{\beta u'(c_{1,h})}{u'(c_0)}(p_1^2 + d_1^2(h)) + \pi_l \frac{\beta u'(c_{1,l})}{u'(c_0)}(p_1^2 + d_1^2(l))\right) \\ 1 &= \frac{1}{2}\frac{\beta u'(c_{1,h})}{u'(c_0)}(2) \qquad (3) \end{aligned}$$

From (3), we can back out the asset pricing kernel for state h:

$$\frac{\beta u'(c_{1,h})}{u'(c_0)} = 1$$

Plugging this result into (2), we get the pricing kernel for state l:

$$\frac{\beta u'(c_{1,l})}{u'(c_0)} = 1$$

Thus, we have $u'(c_{1,h}) = u'(c_{1,l})$, which implies $c_{1,h} = c_{1,l}$ (provided that marginal utility is decreasing). This is not a surprising result, because markets are complete the household chooses a portfolio that eliminates all uncertainty. Moreover, $\beta u'(c_1) = u'(c_0)$ implies that $c_0 > c_{1,h} = c_{1,l}$ (provided that marginal utility is decreasing).