Problem Set I

Macroeconomics II

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1 CIES/CRRA Utility Functions

Suppose that a household lives for two periods ($t = 0, 1$) and maximizes lifetime utility given by:

$$u(c_0) + \beta u(c_1) \quad \text{with} \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma} \qquad \sigma > 0 \text{ and } \sigma \neq 1 \tag{1}$$

1. The *coefficient of relative risk aversion* is defined as $-\frac{u''(c_t)}{u'(c_t)} c_t$. Derive the coefficient of relative risk aversion.

2. Now use the more general CRRA-utility function: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. Show that $\lim_{\sigma \to 1} u(c) = ln(c)$. *Hint:* Use *l'Hôpital's rule*: If $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = 0$ or $\pm\infty$, then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$

3. Suppose the household receives some income w_0 at date 0 and receives no income afterwards. The intertemporal budget constraint is then given by:

$$c_0 + \frac{c_1}{R} = w_0 \qquad w_0 > 0, R > 1$$

   Write down the Lagrangian for the households' optimization problem and solve for the optimal consumption at date 1 relative to date 0 $\left(\frac{c_1^*}{c_0^*}\right)$.

4. Derive the intertemporal elasticity of substitution.

   Hints: The *intertemporal elasticity of substitution* denotes the elasticity of optimal consumption $\frac{c_1^*}{c_0^*}$ with respect to changes in the interest rate R. Recall that the elasticity of a function $f(x)$ with respect to x equals:

$$\varepsilon_x = \frac{x}{f(x)} f'(x)$$

2 Constant Returns to Scale Production Functions

Suppose firms employ a production function $f(K, L)$ with constant returns to scale.

1. Denote w and r as the market wage and capital rental rate respectively. Suppose factor markets are competitive, such that the wage equals the marginal product of labor ($w = f_L(K, L)$)[1] and the capital rental rate equals the marginal product of capital ($r = f_K(K, L)$). Show that constant returns to scale combined with competitive factor markets imply that firms make zero profits.

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*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

[1] $f_x(x, y)$ denotes the partial derivative of $f(x, y)$ with respect to y.

Hint: Use *Euler's theorem* which says that, if a function $f(x_1, .., x_n)$ is homogenous of degree k, then $\sum_{j=1}^{n} \frac{\partial f(x_1, .., x_n)}{\partial x_j} x_j = k\, f(x_1, .., x_n)$.

2. Write output per worker as a function of the capital-labor ratio $k = \frac{K}{L}$
3. Write both the wage rate and the capital rental rate as functions of k (assuming that w and r equal the marginal product of the respective production factor).
4. What would firm profits be if there were increasing or decreasing returns to scale? (With competitive factor markets).

3 Competitive Equilibrium in an Exchange Economy

Consider an economy with two goods, x and y. There are N households each endowed with $\bar{x}$ of good x and zero of good y, and M households each endowed with zero of good x and $\bar{y}$ of good y. All households have identical preferences $U = u(x) + u(y)$. Utility $u(..)$ is strictly increasing and strictly concave. Furthermore, consumption of both goods is essential, that is, it is never optimal to consume zero of one good. Suppose prices in this economy are expressed in terms of good x (x is the so called "numeraire") so the price of x is 1 and p is the price of good y in terms of good x.

1. Write down the maximization problem of both types of households.
2. Assume CRRA-utility. What is the elasticity of substitution between good x and y with respect to changes in p?
3. Assume log-utility (i.e. $\sigma = 1$). Solve for consumption levels of both types of households.
4. Define the competitive equilibrium in this economy and derive the equilibrium price p.

4 Competitive Equilibrium in a Production Economy

We now consider a static economy with production. There are N households. Each is endowed with $\bar{k}$ units of capital. Households work l_h hours and consume c consumption goods. Their utility is $U = u(c) - v(l_h)$ where $u(\cdot)$ is increasing and strictly concave and $v(\cdot)$ is increasing and weakly convex. There are M firms, each of them produces consumption goods with a constant return to scale technology out of labor and capital: $Y = f(k_f, l_f)$. Markets are competitive. Assume all prices are denoted in terms of the consumption good c (numeraire). Denote r as the real rental rate of capital (in terms of consumption goods) and w as the real wage.

1. Write down the maximization problem of a household. Derive first order conditions for working and consuming.
2. Write down the maximization problem of a firm. Derive first order conditions for labor and capital demand.
3. Define the competitive equilibrium in this economy.
4. Use $u(c) = ln(c)$, $v(l_h) = l_h$ and $f(k_f, l_f) = k^{\alpha} l^{1-\alpha}$. Derive the equilibrium labor supply, real wage and rental rate of capital.