Problem Set I

Macroeconomics II

Solutions

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1 Solution: CIES/CRRA Utility Functions

Suppose that a household lives for two periods ($t = 0, 1$) and maximizes lifetime utility given by:

$$u(c_0) + \beta u(c_1) \quad \textit{with} \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma} \qquad \sigma > 0 \;\; \textit{and} \;\; \sigma \neq 1 \tag{1}$$

1. *The* coefficient of relative risk aversion *is defined as* $-\frac{u''(c_t)}{u'(c_t)} c_t$. *Derive the coefficient of relative risk aversion.*

   Solution:

   The measure of relative risk aversion (RRA) according to the economists K.J. Arrow and J.W. Pratt is defined as
   $$\text{RRA} = -\frac{u''(c_t)c_t}{u'(c_t)}.$$

   For the felicity function given in this task, the first and second derivatives are given by

   $$u'(c_t) = (1-\sigma)c_t^{-\sigma}\frac{1}{1-\sigma} = c_t^{-\sigma}$$
   $$u''(c_t) = -\sigma c_t^{-\sigma-1}$$

   The RRA is
   $$\text{RRA} = -\frac{u''(c_t)c_t}{u'(c_t)} = -\frac{-\sigma c_t^{-\sigma-1}c_t}{c_t^{-\sigma}} = \sigma.$$

   Thus, the function given in this exercise features constant relative risk aversion (CRRA).

2. *Now use the more general CRRA-utility function:* $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. *Show that* $\lim_{\sigma\to 1} u(c) = ln(c)$. *Hint: Use* l'Hôpital's rule*: If* $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ *or* $\pm\infty$, *then* $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$

   Solution:

   With $\sigma \to 1$, both the numerator and the denominator go to zero. Therefore, we can apply l'Hôpital's rule: If

   $$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0, \text{then} \lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

Applying this rule we have

$$\lim_{\sigma\to 1}\frac{c_t^{1-\sigma}-1}{1-\sigma}=\lim_{\sigma\to 1}\frac{e^{(1-\sigma)ln(c_t)}-1}{1-\sigma}=$$
$$\lim_{\sigma\to 1}\frac{c_t^{1-\sigma}(-\ln(c_t))}{-1}=\ln(c_t).$$

Here we use the rules of logarithm where $a^x = e^{ln(a^x)} = e^{xln(a)}$ to differentiate with respect to x and then the rule of l'Hôpital in the second equation.

3. *Suppose the household receives some income w_0 at date 0 and receives no income afterwards. The intertemporal budget constraint is then given by:*

$$c_0+\frac{c_1}{R}=w_0 \qquad w_0>0, R>1$$

Write down the Lagrangian for the households' optimization problem and solve for the optimal consumption at date 1 relative to date 0 $\left(\frac{c_1^*}{c_0^*}\right)$.

Solution:

The Lagrangian is given by:

$$\mathcal{L}=u(c_0)+\beta u(c_1)-\lambda\left(c_0+\frac{c_1}{R}-w_0\right)$$

The first-order conditions are given by:

$$\frac{\partial\mathcal{L}}{\partial c_0}=u'(c_0)-\lambda=0 \tag{2}$$
$$\frac{\partial\mathcal{L}}{\partial c_1}=\beta u'(c_1)-\frac{1}{R}\lambda=0 \tag{3}$$

From (2) we get $\lambda = u'(c_0)$. Inserting this into (3) gives $\beta u'(c_1)-\frac{1}{R}u'(c_0)=0$. Hence optimal consumption levels satisfy:

$$\frac{u'(c_0^*)}{u'(c_1^*)}=\left(\frac{c_1^*}{c_0^*}\right)^{\sigma}=\beta R \quad\Rightarrow\quad \frac{c_1^*}{c_0^*}=(\beta R)^{\frac{1}{\sigma}} \tag{4}$$

4. *Derive the intertemporal elasticity of substitution. Hints: The* intertemporal elasticity of substitution *denotes the elasticity of optimal consumption $\frac{c_1^*}{c_0^*}$ with respect to changes in the interest rate R. Recall that the elasticity of a function $f(x)$ with respect to x equals:*

$$\varepsilon_x=\frac{x}{f(x)}f'(x)$$

Solution:

The intertemporal elasticity of substitution (IES) measures the percentage change of the consumption in period $t+1$ relative to the consumption in period t, given a one percent change in the consumption good's price in $t+1$ relative to its price in t:

$$\text{IES}=-\frac{\%\Delta\left(\frac{c_{t+1}}{c_t}\right)}{\%\Delta\left(\frac{p_{c_{t+1}}}{p_{c_t}}\right)}.$$

The first step consists in recognizing that $\%\Delta$ can be replaced by the log-difference:

$$\frac{d\ln(z)}{dz} \approx \frac{1}{z} \Rightarrow d\ln(z) \approx \frac{dz}{z} = \%\Delta z. \tag{5}$$

The next step is to recognize that the relative price equals the marginal rate of substitution:

$$\left(\frac{p_{c_{t+1}}}{p_{c_t}}\right) = \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right).$$

In our case, the marginal rate of substitution is given as follows:

$$\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right) = \left(\frac{\beta c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\right) = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}.$$

Therefore, we get

$$\begin{aligned}\text{IES} &= -\frac{\%\Delta\left(\frac{c_{t+1}}{c_t}\right)}{\%\Delta\left(\frac{p_{c_{t+1}}}{p_{c_t}}\right)} = -\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\ln\left(\frac{p_{c_{t+1}}}{p_{c_t}}\right)} = -\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\ln\left(\beta\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}\right)} \\ &= -\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\ln\beta + d\ln\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}} = -\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{-\sigma d\ln\left(\frac{c_{t+1}}{c_t}\right)} = \frac{1}{\sigma}.\end{aligned}$$

Thus, the function given in this exercise has constant IES, which is given by the inverse of the relative risk aversion σ.

Equivalently we could make use of the definition given in the question, i.e. use the definition of the elasticity $\varepsilon_x = \frac{x}{f(x)}f'(x)$. First, we need to find the components of our function. Recall that we are interested in the response of the consumption path to a change in the price of tomorrow's good. The price of tomorrow's good is given by the interest rate R. The consumption path is given by the solution from (4). It follows that $f(x) = (\beta x)^{\frac{1}{\sigma}}$, where we replaced the interest rate R with x to emphasize that this is the variable of interest. The IES is then given by

$$\begin{aligned}\varepsilon_x &= \frac{x}{f(x)}f'(x) \\ &= \frac{x}{(\beta x)^{\frac{1}{\sigma}}}\beta(\beta x)^{\frac{1}{\sigma}-1}\frac{1}{\sigma} \\ &= \frac{\beta x}{(\beta x)^{\frac{1}{\sigma}}}(\beta x)^{\frac{1}{\sigma}-1}\frac{1}{\sigma} \\ &= \frac{1}{\sigma}\end{aligned}$$

2 Solution: Constant Returns to Scale Production Functions

Suppose firms employ a production function $f(K, L)$ with constant returns to scale.

1. *Denote w and r as the market wage and capital rental rate respectively. Suppose factor markets are competitive, such that the wage equals the marginal product of labor ($w = f_L(K, L)$)[1] and the capital rental rate equals the marginal product of capital ($r = f_K(K, L)$). Show that constant returns to scale combined with competitive factor markets imply that firms make zero profits. Hint:*

[1] $f_x(x, y)$ denotes the partial derivative of $f(y, y)$ with respect to x. Equivalently, we will use $f_1(x, y)$ to denote the partial derivative of $f(x, y)$ with respect to its first argument.

Use Euler's theorem *which says that, if a function* $f(x_1, .., x_n)$ *is homogenous of degree* k*, then* $\sum_{j=1}^{n} \frac{\partial f(x_1, .., x_n)}{\partial x_j} x_j = k\, f(x_1, .., x_n)$.

Solution:

A production function with constant returns to scale has the property that when all input factors are multiplied by α then the output is multiplied by α, too. I.e. the function f is homogeneous of degree 1.

$$f(\alpha K_t, \alpha L_t) = \alpha f(K_t, L_t)$$

In a competitive factor market, capital and labour are rewarded according to the value of their marginal product. We can show this from the maximization problem of the firm:

$$\max_{K,L} f(K, L) - rK - wL$$

$$\Rightarrow f_1(K, L) = r \tag{6}$$

$$f_2(K, L) = w \tag{7}$$

Making use of equation (6) and (7), we can write the profits π of the firm in the following form:

$$\pi = f(K, L) - f_1(K, L)K - f_2(K, L)L \tag{8}$$

From the properties of a CRTS function we know that

$$\alpha f(K_t, L_t) = f(\alpha K_t, \alpha L_t) \tag{9}$$

holds for all α. Differentiating equation (8) with respect to α yields

$$f(K_t, L_t) = f_1(\alpha K_t, \alpha L_t)K_t + f_2(\alpha K_t, \alpha L_t)L_t \tag{10}$$

$$= f_1(K_t, L_t)K_t + f_2(K_t, L_t)L_t. \tag{11}$$

Where we find equation (11) by evaluating equation (10) for $\alpha = 1$. Using the result from equation (11) we find that the profits from equation (8) must be zero.

2. *Write output per worker as a function of the capital-labor ratio* $k = \frac{K}{L}$

   Solution:

   We use $\frac{Y}{L} = y$ and $\frac{K}{L} = k$ as the notation for output per worker and capital per worker. Using the properties of a CRTS function and choosing $\alpha = \frac{1}{L}$ we can write the oput per worker in the following form:

$$y_t = \frac{Y_t}{L_t} = f\left(\frac{K_t}{L_t}, 1\right) = f(k_t, 1) = f(k_t) \tag{12}$$

3. *Write both the wage rate and the capital rental rate as functions of* k *(assuming that* w *and* r *equal the marginal product of the respective production factor).*

   Solution:

   To find the marginal products we multiply equation (12) with L_t and take derivatives with respect to K_t and L_t:

$$
\begin{aligned}
y_t L_t = f(K_t, L_t) = f\left(\frac{K_t}{L_t}, 1\right) L_t & \\
f_1(K_t, L_t) = f_1\left(\frac{K_t}{L_t}, 1\right) L_t \frac{1}{L} = f_1(k_t, 1) & \quad (13) \\
f_2(K_t, L_t) = f\left(\frac{K_t}{L_t}, 1\right) + f_1\left(\frac{K_t}{L_t}, 1\right) L_t \left(-\frac{K}{L^2}\right) = f(k_t, 1) - f_1(k_t, 1) k_t & \quad (14) \\
& \quad (15)
\end{aligned}
$$

From (6) and (7) we know that the rental rate is given by (13) and the wage is given by (14). Example: Cobb-Douglas Production function $f(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$

$$
\begin{aligned}
y_t = \frac{f(K_t, L_t)}{L_t} = \left(\frac{K_t}{L_t}\right)^\alpha = k_t^\alpha \\
f_1(K_t, L_t) = \alpha K_t^{\alpha-1} L_t^{1-\alpha} = \alpha k_t^{\alpha-1} = f_1(k_t, 1) \\
f_2(K_t, L_t) = (1-\alpha) K_t^\alpha L_t^{-\alpha} = (1-\alpha) k^\alpha = f(k_t, 1) - f_1(k_t, 1) k_t
\end{aligned}
$$

4. *What would firm profits be if there were increasing or decreasing returns to scale? (With competitive factor markets).*

   Solution:

   $f(K, L)$ is assumed to be a homogenous function of degree t. This means it satisfies:

   $$f(\alpha K, \alpha L) = \alpha^t f(K, L)$$

   The statement describes how total output changes if the input factors are both multiplied with α. Constant returns to scale means $t = 1$, decreasing returns means $t < 1$ and increasing returns means $t > 1$. The Euler theorem implies:

   $$f_K(K, L)K + f_L(K, L)L = t f(K, L)$$

   and firm profits solve under perfect competition:

   $$\pi = f(K, L) - rK - wL = f(K, L) - f_K(K, L)K - f_L(K, L)L = f(K, L)(1 - t)$$

   So with increasing returns profits are negative and with decreasing returns they are positive.

3 Solution: Competitive Equilibrium in an Exchange Economy

Consider an economy with two goods, x and y. There are N households each endowed with $\bar{x}$ of good x and zero of good y, and M households each endowed with zero of good x and $\bar{y}$ of good y. All households have identical preferences $U = u(x) + u(y)$. Utility $u(\cdot)$ is strictly increasing and strictly concave. Furthermore, consumption of both goods is essential, that is, it is never optimal to consume zero of one good. Suppose prices in this economy are expressed in terms of good x (x is the so called "numeraire") so the price of x is 1 and p is the price of good y in terms of good x.

1. *Write down the maximization problem of both types of households.*

   Solution:

The first households solve:

$$\max_{x_N, y_N} \; u(x_N) + u(y_N) \quad \text{subject to} \quad x_N + p y_N \leq \bar{x}$$

We can write down the Lagrangian:

$$\mathcal{L} = u(x_N) + u(y_N) - \lambda(x_N + p y_N - \bar{x})$$

The first-order conditions are given by:

$$\begin{aligned} u'(x_N) &= \lambda > 0 \\ u'(y_N) &= \lambda p \end{aligned}$$

The Lagrangian for the second households is:

$$\mathcal{L} = u(x_M) + u(y_M) - \lambda(x_M + p y_M - \bar{y} p)$$

and the first-order conditions are given by:

$$\begin{aligned} u'(x_M) &= \lambda > 0 \\ u'(y_M) &= \lambda p \end{aligned}$$

in both cases the first order conditions imply optimal consumption solves

$$\frac{u'(y)}{u'(x)} = p \tag{16}$$

2. *Assume CRRA-utility. What is the elasticity of substitution between good x and y with respect to changes in p?*

Solution:

under CRRA-utility (16) reads:

$$\frac{u'(y)}{u'(x)} = \left(\frac{x}{y}\right)^{\sigma} = p \quad \leftrightarrow \quad \frac{x}{y} = p^{1/\sigma}$$

And the elasticity of substitution between x and y with respect to p is:

$$\frac{p}{(x/y)} \frac{\partial(x/y)}{\partial p} = \frac{p}{p^{1/\sigma}} \frac{1}{\sigma} p^{1/\sigma - 1} = \frac{1}{\sigma} \tag{17}$$

3. *Assume log-utility (i.e. $\sigma = 1$). Solve for consumption levels of both types of households.*

Solution:

Under log-utility ($\sigma = 1$) (16) implies:

$$x_i = p y_i \quad \text{for} \quad i = \{N, M\} \tag{18}$$

Combining this with the binding budget constraints of both households ($x_N + p y_N = \bar{x}$ and $x_M + p y_M = \bar{y} p$) we get for the first households:

$$x_N = \frac{\bar{x}}{2} \tag{19}$$
$$y_N = \frac{\bar{x}}{2p} \tag{20}$$

And for the second households

$$x_M = \frac{\bar{y}p}{2} \tag{21}$$
$$y_M = \frac{\bar{y}}{2} \tag{22}$$

Both households consume half of their endowment themselves and sell the rest to buy the other consumption good.

4. *Define the competitive equilibrium in this economy and derive the equilibrium price p.*

   Solution:

   In general a competitive equilibrium is an allocation and a set of prices such that:

   - all agents (households, firms) behave optimally
   - markets clear (aggr. supply = aggr. demand)

   In this economy optimality means that households of type N consume according to (19) and (20), households of type M consume according to (21) and (22), and market clearing implies:

   - good x: $\underbrace{N\bar{x}}_{\text{supply}} = \underbrace{Nx_N + Mx_M}_{\text{demand}}$
   - good y: $\underbrace{M\bar{y}}_{\text{supply}} = \underbrace{Ny_N + My_M}_{\text{demand}}$

   To get the equilibrium price p^* we insert $x_N = \frac{\bar{x}}{2}$ and $x_M = \frac{\bar{y}p}{2}$ into the market clearing for good x and get:

$$p^* = \frac{N\bar{x}}{M\bar{y}}$$

   Note that we could have used market clearing in the market for good y to get the same result. This follows from *Walras law* which states that in a general equilibrium model if $N-1$ markets clear also the Nth market must clear.

4 Solution: Competitive Equilibrium in a Production Economy

We now consider a static economy with production. There are N households. Each is endowed with $\bar{k}$ units of capital. Households work l_h hours and consume c consumption goods. Their utility is $U = u(c) - v(l_h)$ where $u(\cdot)$ is increasing and strictly concave and $v(\cdot)$ is increasing and weakly convex. There are M firms, each of them produces consumption goods with a constant return to scale technology out of labor and capital: $Y = f(k_f, l_f)$. Markets are competitive. Assume all prices are denoted in terms of the consumption good c (numeraire). Denote r as the real rental rate of capital (in terms of consumption goods) and w as the real wage.

1. *Write down the maximization problem of a household. Derive first order conditions for working and consuming.*

   Solution:

The households in this economy solve:

$$\max_{c,l_H} \; u(c) - v(l_H) \quad \text{subject to} \quad c \leq \bar{k}r + l_H w + \pi$$

π are the profits per household in terms of consumptions goods (households are assumed to own the firms and get any excess profits).

We can write down the Lagrangian:

$$\mathcal{L} = u(c) - v(l_H) - \lambda(c - \bar{k}r - l_H w - \pi)$$

The first-order conditions are given by:

$$u'(c) = \lambda \tag{23}$$

$$\lambda w = v'(l_H) \tag{24}$$

$$c = \bar{k}r + l_H w + \pi \tag{25}$$

Where (25) is simply the budget constraint of the economy. Note that from (23) we find that $\lambda > 0$ which implies that the budget constraint must hold with equality. From (23) and (27) we find the *Intratemporal marginal rate of substitution*

$$\frac{v'(l_H)}{u'(c)} = w \tag{26}$$

2. *Write down the maximization problem of a firm. Derive first order conditions for labor and capital demand.*

   Solution:

   The firms in this economy solve:

$$\max_{k_f,l_f} \; \pi = f(k_f, l_f) - k_f r - l_f w$$

   which yields the usual first-order-conditions:

$$f_{l_f}(k_f, l_f) = w \tag{27}$$

$$f_{k_f}(k_f, l_f) = r \tag{28}$$

   From the properties of constant returns to scale and a competitive economy we know that$\pi = 0$.

3. *Define the competitive equilibrium in this economy.*

   Solution:

   A competitive equilibrium is defined by a set of prices such that all agents behave optimally and markets clear.

   Optimality implies:

   - households choose c and l_H optimally according to (26) and(25)
   - firms choose l_f and k_f according to (27) and (28)

market clearing implies:

$$\begin{aligned}
&\text{capital market:} && \underbrace{N\bar{k}}_{\text{supply}} = \underbrace{Mk_f}_{\text{demand}} \\
&\text{labor market:} && \underbrace{Nl_H}_{\text{supply}} = \underbrace{Ml_f}_{\text{demand}} \\
&\text{consumption good:} && \underbrace{Mf(k_f, l_f)}_{\text{supply}} = \underbrace{Nc}_{\text{demand}}
\end{aligned}$$

4. *Use $u(c) = ln(c)$, $v(l_h) = l_h$ and $f(k_f, l_f) = k_f^{\alpha} l_f^{1-\alpha}$. Derive the equilibrium labor supply, real wage and rental rate of capital.*

Solution:

Using $u(c) = ln(c)$, $v(l_h) = l_h$ and $f(k_f, l_f) = k_f^{\alpha} l_f^{1-\alpha}$ (26) and (25) yield

$$c = w \tag{29}$$

$$l_H = 1 - \frac{r}{w}\bar{k} \tag{30}$$

where the last line follows from the properties of the firms production function, which implies zero profits due to the constant returns to scale assumption. From the firms optimality conditions in (27) and (28) we find

$$r = \frac{\alpha}{k^{1-\alpha}} \tag{31}$$

$$w = (1-\alpha)k^{\alpha} \tag{32}$$

where $k = \frac{k_f}{l_f}$. Thus the ratio $\frac{r}{w}$ is:

$$\frac{r}{w} = \frac{\alpha}{k^{1-\alpha}} \frac{1}{(1-\alpha)k^{\alpha}} = \frac{\alpha}{1-\alpha}\frac{1}{k}$$

From capital market clearing we know: $k_f^* = N/M\bar{k} = n\bar{k}$ where we define $n = N/M$. Then we can express the labor supply of firms as $l_F = nl_H$. Thus $k = \frac{n\bar{k}}{nl_H} = \frac{\bar{k}}{l_H}$ and we can solve for l_H with (30)

$$\begin{aligned}
l_H &= 1 - \frac{r}{w}\bar{k} \\
&= 1 - \frac{\alpha}{1-\alpha}\frac{l_H}{\bar{k}}\bar{k} \\
&= 1 - \alpha
\end{aligned}$$

Then the equilibrium allocation is:

$$
\begin{aligned}
l_H^* &= 1 - \alpha \\
l_F^* &= n(1 - \alpha) \\
w^* &= (1 - \alpha)\left(\frac{\bar{k}}{1 - \alpha}\right)^{\alpha} = c^* \\
r^* &= \frac{\alpha}{\left(\frac{\bar{k}}{1-\alpha}\right)^{1-\alpha}}
\end{aligned}
$$

Note that we only used capital and labor market clearing to derive this allocation. By *Walras' law* we know that the market for consumption must also clear.