Problem Set III

Macroeconomics II

Solutions

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1 Solution: No-Ponzi-game condition

Consider an infinitely lived household with assets $a_0 < 0$ at time $t = 0$. The household compares different options of servicing the debt, as specified below. For each option, check whether the household satisfies the condition $\lim_{T\to\infty} a_{T+1} \geq 0$ and/or the no-Ponzi-game condition $\lim_{T\to\infty} q_T a_{T+1} \geq 0$,, where $q_T = \frac{1}{R_1 R_2 \cdots R_T}$ and compute the present discounted value of the debt service (discounting at the gross interest rate R). Assume that income is sufficiently high such that each option is feasible.

1. *In period 0, fully repay all debt (i.e., pay $-Ra_0$).*

   Solution:

   Very simple strategy. In period 0 service all the debt and choose $a_1 = 0$. The debt payment (S) is therefore given by $S_t = -a_0 R$ for period $t = 0$ and $S_t = 0, \forall t > 0$, hence $a_1 = a_0 + S_0 = 0$ and $a_t = 0, \forall t > 1$. What about the two conditions:

   - $\lim_{T\to\infty} a_{T+1} \geq 0$: Since $a_t = 0, \forall t > 1$ this is clearly satisfied.
   - $\lim_{T\to\infty} q_T a_{T+1} \geq 0$: Since $a_t = 0, \forall t > 1$ this is also satisfied.

   Net present value of all debt payments is given by $\sum_{t=0}^{\infty} q_t S_t = -a_0 R$.

2. In each period $t \geq 0$, pay $-xa_t$, where $0 < x < R$.

   Solution:

   In this strategy households choose to repay a fraction of the debt in every period $S_t = -xa_t, \forall t$. The assets of the household evolve as follows:

$$\begin{aligned} a_{t+1} &= a_t R_t + S_t \\ &= a_t R - x a_t \\ &= a_t (R - x) \\ &= a_0 (R - x)^{t+1} \end{aligned}$$

   What about the two conditions:

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

- $\lim_{T\to\infty} a_{T+1} \geq 0$:

$$\lim_{T\to\infty} a_{T+1} = \lim_{T\to\infty} a_0(R-x)^{T+1} \overset{?}{>} 0$$

  The condition is only fulfilled if $R - x < 1$.

- $\lim_{T\to\infty} q_T a_{T+1} \geq 0$:

$$\begin{aligned}\lim_{T\to\infty} q_T a_{T+1} &= \lim_{T\to\infty} \frac{1}{R^T} a_0(R-x)^{T+1} \\ &= \lim_{T\to\infty} (R-x)\left(\frac{R-x}{R}\right)^T a_0 = 0\end{aligned}$$

  The no-Ponzi-game condition is always fulfilled since $\frac{R-x}{R} < 1$

Net present value of all debt payments is given by $\sum_{t=0}^{\infty} q_t S_t$.

$$\begin{aligned}\sum_{t=0}^{\infty} q_t S_t &= \sum_{t=0}^{\infty} \frac{1}{R^t} \underbrace{(-xa_0)(R-x)^t}_{=S_t} \\ &= -xa_0 \sum_{t=0}^{\infty} \left(\frac{R-x}{R}\right)^t \\ &= -xa_0 \frac{1}{1-\frac{R-x}{R}} \\ &= -a_0 R\end{aligned}$$

The net present value of the debt service with the second strategy is therefore equivalent to strategy number one.

3. *In each period $t \geq 0$, don't pay anything.*

   Solution: In the last strategy households choose to repay none of the debt in every period $S_t = 0, \forall t$. The assets of the household evolve as follows are therefore decreasing as the interest rates are continued to be payed with additional debt and $a_t = R^t a_0$. The net present value of the debt service is therefore equal to zero since no debt is ever paid. What about the two conditions:

   - $\lim_{T\to\infty} a_{T+1} \geq 0$:

$$\lim_{T\to\infty} a_{T+1} = R^{T+1} a_0 \to -\infty$$

     The first condition is clearly violated as the debt keeps piling up.

   - $\lim_{T\to\infty} q_T a_{T+1} \geq 0$:

$$\lim_{T\to\infty} q_T a_{T+1} = \frac{R^{T+1}}{R^T} a_0 = Ra_0 < 0$$

     The no-Ponzi-game condition is also violated. In fact, this strategy is what we call a Ponzi-game because the households try to repay the debt by paying their old debt with new debt plus additional debt for the interest payments. It's called after Charles Ponzi an Italian who used the Ponzi scheme in the US to enrich himself.

2 Solution: Intertemporal budget constraint with infinite horizon

Show that the no-Ponzi-game condition together with the dynamic budget constraints implies the intertemporal budget constraint

$$a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) \geq 0.$$

Solution:

Remember that in the finite case the intertemporal budget constraint is given by:

$$a_0 R_0 + \sum_{t=0}^{T} q_t(w_t - c_t) = q_T a_{T+1}$$

with the terminal condition $q_T a_{T+1} \geq 0$. In the infinite horizon model the IBC therefore becomes

$$a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) = \lim_{T\to\infty} q_T a_{T+1}.$$

Combined with the no-Ponzi-game condition the IBC is therefore given by

$$a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) = \lim_{T\to\infty} q_T a_{T+1} \geq 0$$
$$a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) \geq 0$$

3 Solution: Non-Geometric Discounting and Time-Consistency

Consider the following three period model, $t = 0, 1, 2$. At date $t = 0$ preferences are given by $u(c_0) + \beta(u(c_1) + u(c_2))$, but at time $t = 1$ preferences are given by $u(c_1) + \beta u(c_2)$. For simplicity, assume that $w_t = w$ and $R_t = 1, t = 0, 1, 2$.

Set up the problem of a household under commitment at $t = 0$ and the problem of a household that re-optimizes at $t = 1$. Derive the Euler equations of the two problems and show that the household that re-optimizes every period would choose a different consumption plan.

Solution:

1. Commitment: A household with commitment will solve the problem at $t = 0$ and then stay on that path forever after

$$\max_{c_0,c_1,c_2} u(c_0) + \beta(u(c_1) + u(c_2))$$
$$\text{s.t. } c_0 + c_1 + c_2 = 3w + a_0$$

with the first order conditions given by

$$c_0 : u'(c_0) = \lambda,$$
$$c_1 : \beta u'(c_1) = \lambda,$$
$$c_2 : \beta u'(c_2) = \lambda.$$

The euler equations are then given by

$$u'(c_0) = \beta u'(c_1) \tag{1}$$
$$u'(c_1) = u'(c_2) \tag{2}$$

Since the consumer does not reoptimize at period $t = 1$, we find that $c_1 = c_2$ (given that $u''(\cdot) < 0$.

2. No Commitment: A household without commitment will reoptimize the consumption choice in every period given its utility function and assets. In period $t = 1$ it will therefore solve the following problem

$$\max_{c_1, c_2} u(c_1) + \beta u(c_2)$$
$$\text{s.t. } c_1 + c_2 = 2w + a_1$$

which gives the euler equation

$$u'(c_1) = \beta u'(c_2).$$

This yields $c_1 > c_2$. Note that this is not optimal from the consumers perspective at time $t = 0$. The consumer at time $t = 0$ anticipates the behaviour of its future self and optimizes the consumption path given the future consumption choice

$$\max_{c_0, c_1, c_2} u(c_0) + \beta(u(c_1) + u(c_2))$$
$$\text{s.t. } c_0 + c_1 + c_2 = 3w + a_0$$
$$u'(c_1) = \beta u'(c_2)$$

Note that the second constraint is simply the euler equation chosen by the household at $t = 1$. The first order conditions are

$$c_0 : u'(c_0) = \lambda_1,$$
$$c_1 : \beta u'(c_1) = \lambda_1 + \lambda_2 u''(c_1),$$
$$c_2 : \beta u'(c_2) = \lambda_1 - \lambda_2 \beta u''(c_2).$$

With λ_2 given by

$$\beta u'(c_1) - \lambda_2 u''(c_1) = \beta u'(c_2) + \lambda_2 \beta u''(c_2)$$
$$\lambda_2 = \beta \frac{u'(c_1) - u'(c_2)}{u''(c_1) + \beta u''(c_2)}.$$

The euler equations are then given by

$$u'(c_0) = \beta u'(c_1) - \beta \frac{u'(c_1) - u'(c_2)}{u''(c_1) + \beta u''(c_2)} u''(c_1) \tag{3}$$
$$u'(c_1) = \beta u'(c_2). \tag{4}$$

Comparing the euler equations (2) with (4) we see that without commitment $c_2 < c_1$ where as with commitment $c_1 = c_2$. Equivalently, because $u''(c_1) < 0$ and $\lambda_2 > 0$, we see from (3) that without commitment $u'(c_0) > \beta u'(c_1)$. It follows that $\frac{c_1}{c_0}$ is larger without commitment, meaning that the consumer in period $t = 0$ is consuming less in the initial period because she anticipates that the consumer in period $t = 1$

will choose $c_1 > c_2$. She therefore reduces c_0 to increase consumption in period $t = 2$ where the marginal benefit will be larger from todays perspective.

4 Solution: Dynamic programming

Let $V_t(a_t)$ denote the maximal utility which the household can achieve from period t on if it enters the period with assets a_t.

1. *Show that*

$$
\begin{aligned}
V_t(a_t) = &\max_{\{c_s, a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s) \\
&\textit{s.t. } a_{s+1} = a_s R_s + w_s - c_s \textit{ for } s = t, t+1, \ldots, T \\
&\quad a_t \textit{ given} \\
&\quad a_{T+1} \geq 0
\end{aligned}
$$

can be rewritten as

$$
\begin{aligned}
V_t(a_t) = &\max_{c_t, a_{t+1}} [u(c_t) + \beta V_{t+1}(a_{t+1})] \\
&\textit{s.t. } a_{t+1} = a_t R_t + w_t - c_t \\
&\quad a_t \textit{ given.}
\end{aligned}
$$

Solution:

For ease of notation let $\mathcal{C}$ denote the set of dynamic budget constraints at date $t+1$ and later as well as the terminal condition $a_{T+1} \geq 0$

$$
\begin{aligned}
V_t(a_t) &= \max_{\{c_s, a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s) \text{ s.t. DBC}_t, \mathcal{C}, a_t \text{ given} \\
&= \max_{c_t, a_{t+1}} u(c_t) + \left(\max_{\{c_s, a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-t} u(c_s) \text{ s.t. } \mathcal{C}, a_{t+1} \text{ given} \right), \text{ s.t. DBC}_t, a_t \text{ given} \\
&= \max_{c_t, a_{t+1}} u(c_t) + \beta \left(\underbrace{\max_{\{c_s, a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-(t+1)} u(c_s) \text{ s.t. } \mathcal{C}, a_{t+1} \text{ given}}_{=V_{t+1}(a_{t+1})} \right), \text{ s.t. DBC}_t, a_t \text{ given} \\
V_t(a_t) &= \max_{c_t, a_{t+1}} u(c_t) + \beta V_{t+1}(a_{t+1}), \text{ s.t. DBC}_t, a_t \text{ given}
\end{aligned}
$$

Now consider the Bellman equation

$$
V_t(a_t) = \max_{a_{t+1}} \left[u(a_t R_t + w_t - a_{t+1}) + \beta V_{t+1}(a_{t+1}) \right].
$$

Note that $V_{T+1}(a_{T+1}) = 0$, implying the optimal choice $a^*_{T+1} = 0$.

2. *By backward induction, solve for the value functions $V_T(a_T)$ and $V_{T-1}(a_{T-1})$, and for the policy functions $a_T(a_{T-1})$ and $a_{T-1}(a_{T-2})$, given the assumptions that $u(c_t) = \ln(c_t)$ and $w_t = 0$ for all t.*

Solution:

$$\begin{aligned} V_T(a_T) &= \max_{a_{T+1}} u(a_T R_T + \underbrace{w_T}_{=0} - a_{T+1}) + \beta \underbrace{V_{T+1}(a_{T+1})}_{=0, \forall a_{T+1}} \\ &= \max_{a_{T+1}} u(a_T R_T - a_{T+1}) \end{aligned}$$

It follows that the optimal policy function and value function at T are given as follows:

$$\begin{aligned} a^*_{T+1}(a_T) &= 0 \\ V_T(a_T) &= u(a_T R_T) = \ln(a_T R_T) \end{aligned}$$

We can continue by backwards induction:

$$\begin{aligned} V_{T-1}(a_T) &= \max_{a_T} u(a_{T-1}R_{T-1} - a_T) + \beta V_T(a_T) \\ &\max_{a_T} \ln(a_{T-1}R_{T-1} - a_T) + \beta \ln(a_T R_T) \end{aligned}$$

This is a problem that we can solve for. Let's start by finding the first order conditions wrt a_T:

$$\begin{aligned} V'_{T-1}(a_T) = \frac{-1}{a_{T-1}R_{T-1} - a_T} + \beta \frac{R_T}{a_T R_T} &\stackrel{!}{=} 0 \\ a_T &= \beta(a_{T-1}R_{T-1} - a_T) \\ a^*_T(a_{T-1}) &= \frac{\beta}{1+\beta} a_{T-1}R_{T-1} \end{aligned}$$

The value function is then given by:

$$\begin{aligned} V_{T-1}(a_T) &= \max_{a_T} \ln(a_{T-1}R_{T-1} - a_T) + \beta \ln(a_T R_T) \\ &= \ln\left(a_{T-1}R_{T-1} - \frac{\beta}{1+\beta} a_{T-1}R_{T-1}\right) + \beta \ln\left(\frac{\beta}{1+\beta} a_{T-1}R_{T-1}R_T\right) \\ &= \ln\left(\frac{1}{1+\beta} a_{T-1}R_{T-1}\right) + \beta \ln\left(\frac{\beta}{1+\beta} a_{T-1}R_{T-1}R_T\right) \\ &= (1+\beta)\ln(a_{T-1}) + \varphi \end{aligned}$$

where $\varphi = \ln\left(\frac{1}{1+\beta} R_{T-1}\right) + \beta \ln\left(\frac{\beta}{1+\beta} R_{T-1}R_T\right)$ collects all terms independent of a_{T-1}. Using the result for $V_{T-1}(a_T)$ we can proceed the same way to find the policy function for $a_{T-1}(a_{T-2})$:

$$\begin{aligned} V_{T-2}(a_{T-1}) &= \max_{a_{T-1}} \ln(a_{T-2}R_{T-2} - a_{T-1}) + \beta(1+\beta)\ln(a_{T-1}) + \beta\varphi \\ V'_{T-2}(a_{T-1}) &= \frac{-1}{a_{T-2}R_{T-2} - a_{T-1}} + \frac{\beta(1+\beta)}{a_{T-1}} \stackrel{!}{=} 0 \\ a_{T-1} &= \beta(1+\beta)(a_{T-2}R_{T-2} - a_{T-1}) \\ a^*_{T-1}(a_{T-2}) &= \frac{\beta(1+\beta)}{1+(\beta(1+\beta))} a_{T-2}R_{T-2} \end{aligned}$$

3. *Using the Bellman equation and the envelope theorem, derive the Euler equation.*

Solution:

Different ways to approach this problem, one way is to use the DBC$_t$ to substitute for c_t:

$$V_t(a_t) = \max_{a_{t+1}} u(a_t R_t + w_t - a_{t+1}) + \beta V_{t+1}(a_{t+1})$$

Take the first order condition with respect to a_{t+1}:

$$\begin{aligned} u'(a_t R_t + w_t - a_{t+1})(-1) + \beta V'_{t+1}(a_{t+1}) &\overset{!}{=} 0 \\ u'(\underbrace{c_t}_{=a_t R_t + w_t - a_{t+1}}) &= \beta V'_{t+1}(a_{t+1}) \end{aligned} \tag{5}$$

Revision envelope theorem: Differentiate the value function $V_t(a_t)$ with respect to a_t

$$\begin{aligned} V'_t(a_t) &= u'(\underbrace{c_t}_{=a_t R_t + w_t - a_{t+1}})(R_t - \frac{\partial a_{t+1}}{a_t}) + \beta V'_{t+1}(a_{t+1})\frac{\partial a_{t+1}}{a_t} \\ &= u'(c_t)R_t + \frac{\partial a_{t+1}}{a_t}\left(\beta V'_{t+1}(a_{t+1}) - u'(c_t)\right) \end{aligned}$$

Since the value function is defined by the optimal choice for a_{t+1} we can make use of the result in (5) and we find

$$\begin{aligned} V'_t(a_t) &= u'(c_t)R_t + \frac{\partial a_{t+1}}{a_t}\left(\underbrace{\beta V'_{t+1}(a^*_{t+1}) - u'(c^*_t)}_{=0}\right) \\ &= u'(c_t)R_t \end{aligned}$$

Use the envelope theorem to find $V'_{t+1}(a_{t+1})$

$$\begin{aligned} V'_t(a_t) &= u'(c_t)R_t, \forall t \\ \Rightarrow V'_{t+1}(a_{t+1}) &= u'(c_{t+1})R_{t+1} \end{aligned}$$

Combining the result from the envelope theorem with the result from (5) we find the euler equation:

$$u'(c_t) = \beta R_{t+1} u'(c_{t+1})$$

5 Solution: Dynamic programming with infinite horizon: guess and verify

Assume that $u(c_t) = \ln(c_t)$, $w_t = 0$ and $R_t = R$ for all t. Thus the Bellman equation is given by

$$V(a) = \max_{a_+} \ln(aR - a_+) + \beta V(a_+).$$

Solve for the value function $V(a)$ and for the policy function $g(a)$ in the infinite horizon case using a guess and verify approach. An educated guess for the two functions is

$$\begin{aligned} V(a) &= F\ln a + G \\ g(a) &= HaR, \end{aligned}$$

where F, G and H are unknown coefficients. Proceed as follows:

1. *Using the Bellman equation, show that given the guess for $V(a)$, the policy function is indeed of the form $g(a) = HaR$. Solve for H as a function of β, F and G.*

   Solution:

Using the suggested guess the bellman equation becomes

$$\begin{aligned} V(a) &= \max_{a_+} \ln(aR - a_+) + \beta V(a_+) \\ F\ln a + G &= \max_{a_+} \ln(aR - a_+) + \beta(F\ln a_+ + G) \end{aligned}$$

We can find the policy function $g(a)$ by taking the first derivative with respect to a_+:

$$\begin{aligned} \frac{-1}{aR - a_+} + \beta\frac{F}{a_+} &\overset{!}{=} 0 \\ a_+ &= \beta F(aR - a_+) \\ a_+ &= \frac{\beta F}{1+\beta F}aR. \end{aligned} \tag{6}$$

Conditional on our guess for the value function we verified that $g(a) = HaR$, where $H = \frac{\beta F}{1+\beta F}$.

2. *Using the Bellman and your result from a., use the method of undetermined coefficients to solve for the coefficients F, G and H.*

   Solution:

   To use the method of undetermined coefficients we make use of the result from (6) and plug it into our guess for the value function:

$$\begin{aligned} V(a) &= \max_{a_+} \ln(aR - a_+) + \beta V(a_+) \\ F\ln a + G &= \ln\left(aR - \frac{\beta F}{1+\beta F}aR\right) + \beta\left[F\ln\left(\frac{\beta F}{1+\beta F}aR\right) + G\right] \\ &= \ln\left(\left[1 - \frac{\beta F}{1+\beta F}\right]aR\right) + \beta\left[F\ln\left(\frac{\beta F}{1+\beta F}aR\right) + G\right] \\ &= \ln\left(\frac{1}{1+\beta F}\right) + \ln(a) + \ln(R) + \beta F\ln\left(\frac{\beta F}{1+\beta F}\right) + \beta F\ln(a) + \beta F\ln(R) + \beta G \end{aligned}$$

   where the $\max_{a_+}$ is dropped in the secon line because (conditional on our guess is correct) $g(a)$ is maximizing the value function. We now collect terms depending on a to find the coefficient F and collect terms independent of a to find the coefficient G:

$$\begin{aligned} F\ln a + G &= \ln\left(\frac{1}{1+\beta F}\right) + \ln(a) + \ln(R) + \beta F\ln\left(\frac{\beta F}{1+\beta F}\right) + \beta F\ln(a) + \beta F\ln(R) + \beta G \\ \underbrace{F\ln a}_{1.\to F} + \underbrace{G}_{2.\to G} &= \underbrace{(1+\beta F)\ln(a)}_{1.\to F} + \underbrace{\ln\left(\frac{1}{1+\beta F}\right) + +\beta F\ln\left(\frac{\beta F}{1+\beta F}\right) + (1+\beta F)\ln(R) + \beta G}_{2.\to G} \end{aligned} \tag{7}$$

   First we can find the coefficient F:

$$\begin{aligned} F &= (1+\beta F) \\ F &= \frac{1}{1-\beta} \end{aligned} \tag{8}$$

Now we can use (8) in (7) to find the coefficient G

$$\begin{aligned}
G &= \ln\left(\frac{1}{1+\beta F}\right) + \beta F \ln\left(\frac{\beta F}{1+\beta F}\right) + (1+\beta F)\ln(R) + \beta G \\
(1-\beta)G &= \ln\left(\frac{1}{1+\beta\frac{1}{1-\beta}}\right) + \beta\frac{1}{1-\beta}\ln\left(\frac{\beta\frac{1}{1-\beta}}{1+\beta\frac{1}{1-\beta}}\right) + \left(1+\beta\frac{1}{1-\beta}\right)\ln(R) \\
&= \ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta) + \left(\frac{1}{1-\beta}\right)\ln(R) \\
G &= \frac{1}{1-\beta}\left[\ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta) + \left(\frac{1}{1-\beta}\right)\ln(R)\right] \qquad (9)
\end{aligned}$$

The only thing that remains to be done is to find a solution for H by using (8)

$$\begin{aligned}
H &= \frac{\beta F}{1+\beta F} \\
&= \frac{\beta\frac{1}{1-\beta}}{1+\beta\frac{1}{1-\beta}} \\
H &= \beta \qquad (10)
\end{aligned}$$

The results from (8), (9) and (10) verify our initial guess and we found a closed form solution for the value function and the policy function.