Problem Set IV

Macroeconomics II

Solutions

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1 Solution: General Equilibrium Conditions

Consider a neoclassical growth model (Ramsey model) with infinitely lived agents. Households have utility such that $u'(\cdot) > 0$ and $u''(\cdot) > 0$ and firms have a constant return to scale production function.

1. *State the conditions that arise from*

   (a) *utility maximization of households,*

   Solution:

   - Euler equation: $u'(c_t) = \beta R_{t+1} u'(c_{t+1}$
   - DBC: $a_{t+1} = a_t R_t + w_t - c_t + \pi_t$
   - There is no borrowing between homogeneous households
   - Physical capital is the only asset: $\Rightarrow a_t = k_t$
   - Gross return: $R_t = 1 + r_t - \delta$, where r_t is the rental rate of capital and δ the depreciation rate

   From this we find the three conditions that arise from the households problem:

   $$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) \quad (1)$$

   $$k_{t+1} = k_t(1 + r_{t+1} - \delta) + w_t - c_t + \pi_t \quad (2)$$

   $$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 \quad (3)$$

   where (1) is the euler equation, (2) the law of motion of capital and (3) the transversality condition of the household.

   (b) *profit maximization of firms*

   Solution:

   The firm simply maximizes its profit function choosing the input factor needed for production: $\max_{K_t,L_t} f(K_t, L_t) - w_t L_t - r_t K_t$. This yields equilibrium factory payments plus profits:

   $$w_t = f_L(K_t, L_t) \quad (4)$$

   $$r_t = f_K(K_t, L_t) \quad (5)$$

   $$\pi_t = f(K_t, L_t) - w_t L_t - r_t K_t \quad (6)$$

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

(c) *feasibility.*

Solution:

From the market clearing condition we can pin down L_t and K_t:

$$L_t = 1 \tag{7}$$
$$K_t = k_t \tag{8}$$

2. *Combine these conditions to derive the two core equations that characterize the representative agent model.*

Solution:

From (2), (4), (5), (6), (7) and (8) we can find the resource constraint of the economy:

$$\begin{aligned} k_{t+1} &\overset{(2)}{=} k_t(1 + r_t - \delta) + w_t - c_t + \pi_t \\ &\overset{(6)}{=} k_t(1 + r_t - \delta) + w_t - c_t + f(K_t, L_t) - w_t L_t - r_t K_t \\ &\overset{(7)\&(8)}{=} k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - w_t - r_t k_t \\ k_{t+1} &= k_t(1 - \delta) + f(k_t, 1) - c_t \end{aligned} \tag{9}$$

Equivalently, we could write (9) as $f(k_t, 1) = c_t + i_t$ where $i_t = k_{t+1} - (1 - \delta)k_t$ is the investment in capital.

From (1), (5), (7) and (8) we find the euler equation of the represantative household:

$$\begin{aligned} u'(c_t) &\overset{(1)}{=} \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) \\ &\overset{(5)}{=} \beta(1 + f_K(K_{t+1}, L_{t+1}) - \delta)u'(c_{t+1}) \\ u'(c_t) &\overset{(7)\&(8)}{=} \beta(1 + f_k(k_{t+1}, 1) - \delta)u'(c_{t+1}) \end{aligned} \tag{10}$$

The general equilibrium is then defined by the inital k_0 together with (9), (10) and the transversality condition from (3). For a given k_0 the choice for c_0 pins down k_1 through the resource constraint, which pins down c_1 through the euler equation, and so on. This yields $\{c_t, k_{t+1}\}_{t=0}^{\infty}$, which pins down the remaining variables $r_t, w_t, \ldots$. However, note that the initial choice for c_0 is not free but must satisfy the transversality condition from (3).

2 Solution: General Equilibrium: Steady state

Consider the infinitely lived representative agent model analyzed in 1. Suppose that utility is of the CRRA type,

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma},$$

and production is characterized by a Cobb-Douglas function,

$$f(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha},$$

or, in per capita terms,

$$f(k_t, 1) = k_t^{\alpha}.$$

1. *Derive and plot the steady state resource constraint and the steady state Euler equation (with k on the horizontal axis of the diagram and c on the vertical axis).*

Solution:

- Household: Chooses consumption such that life-time utility is maximized:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } k_{t+1} = (1 + r_t - \delta)k_t - c_t + w_t + \pi_t$$

  This yields the households euler equation plus transversality condition:

$$u'(c_t) = \beta(1 + r_t - \delta)u'(c_{t+1})$$
$$\lim_{T\to\infty} \beta^T u'(c_T)k_{t+1} \geq 0$$

- Firm: maximizes profits:

$$\max_{K_t, L_t} f(K_t, L_t) - r_t K_t - w_t L_t$$

  This yields the factor payments:

$$r_t = f_K(K_t, L_t)$$
$$w_t = f_L(K_t, L_t)$$

Together with the market clearing conditions we find the resource constraint and the euler equation

$$k_{t+1} = (1 - \delta)k_t + f(k_t, 1) - c_t$$
$$u'(c_t) = \beta(1 + f_k(k_{t+1}, 1) - \delta)u'(c_{t+1})$$

Evaluated at the steady state values these conditions simplify to the following to equations:

$$\bar{k} = (1 - \delta)\bar{k} + f(\bar{k}, 1) - \bar{c}$$
$$\delta\bar{k} = f(\bar{k}, 1) - \bar{c}$$
$$\bar{c} = f(\bar{k}, 1) - \delta\bar{k} \tag{11}$$
$$u'(\bar{c}) = \beta(1 + f_k(\bar{k}, 1) - \delta)u'(\bar{c})$$
$$1 = \beta(1 + f_k(\bar{k}, 1) - \delta) \tag{12}$$

For the production function and felicity function given in the question (11) and (12) become

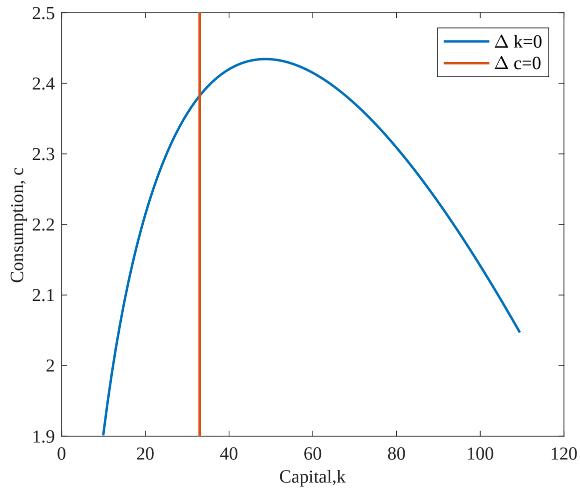
$$\bar{c} = \bar{k}^{\alpha} - \delta\bar{k},$$
$$1 = \beta(1 + \alpha\bar{k}^{\alpha-1} - \delta).$$

The plot of the two nullclines is shown in figure 1.

2. *Solve for the steady state values of consumption and capital (the* modified-golden-rule *capital stock).*

   Solution:

Figure 1: Nullclines



From (12) we solve for the steady state capital k^{ss}

$$
\begin{aligned}
1 &= \beta(1 + f_k(k^{ss}, 1) - \delta) \\
1 &= \beta(1 + \alpha(k^{ss})^{\alpha-1} - \delta) \\
k^{ss} &= \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \qquad (13)
\end{aligned}
$$

We can use the result from (13) together with (11) to find the steady state consumption level:

$$
\begin{aligned}
c^{ss} &= (k^{ss})^{\alpha} - \delta k^{ss} \\
c^{ss} &= \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \qquad (14)
\end{aligned}
$$

3. *Derive the* golden-rule *capital stock. Show that the modified-golden-rule capital stock is necessarily smaller than the golden-rule capital stock.*

Solution:

The golden rule capital stock is defined as the capital stock that maximizes steady state consumption:

$$
\begin{aligned}
k^{gr} &= \arg\max_k c^{ss} \\
&= \arg\max_k f(k) - \delta k
\end{aligned}
$$

Therefore $k^{\text{gr}} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. Now let's compare the two capital levels:

$$\begin{aligned}
k^{\text{gr}} &\overset{?}{>} k^{\text{ss}} \\
\left(\frac{\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} &\overset{?}{>} \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \\
\left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{1-\alpha}} &\overset{?}{>} \left(\frac{\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \\
\frac{1-\beta(1-\delta)}{\beta} &\overset{?}{>} \delta \\
1-\beta(1-\delta) &\overset{?}{>} \beta\delta \\
1 &> \beta
\end{aligned}$$

Equivalently we would find the same conclusion by realising that $f_k(k^{\text{ss}}) = \delta + \frac{1}{\beta} - 1 > \delta = f_k(k^{\text{gr}})$.

4. Why *does steady state consumption fall short of consumption at the golden-rule capital stock (although the equilibrium is Pareto efficient)?*

   Solution

   The consumer does not maximize lifetime consumption but rather lifetime utility. Because future consumption is discounted at rate β the consumer would be better of, if she consumed some of the extra capital rather then investing it. This will increase her current consumption at the cost of higher future consumption, which is optimal from a utility perspective.

   Let us show that the golden rule capital stock would not be optimal. Assume that the initial capital level would be $\bar{k}$, with $k^{\text{ss}} < \bar{k} \leq k^{\text{gr}}$. For this to be the optimal choice it must be that there is no strategy that could improve the agents utility. Let us compare two different strategies: The first strategy is that the consumer stays at the given capital level forever, i.e $k_t = \bar{k}, \forall t$.

$$\begin{aligned}
k_t &= \bar{k}, \forall t, \text{ with } k^{\text{ss}} < \bar{k} \leq k^{\text{gr}} \\
c_t &= \bar{k}^{\alpha} - \delta\bar{k}, \forall t
\end{aligned}$$

   The second strategy starts with $k_0 = \bar{k}$ but then chooses a capital level that is smaller by $\epsilon > 0$

$$\begin{aligned}
k_t &= \bar{k} - \epsilon, \text{ for } t \geq 1 \\
c_0 &= \bar{k}^{\alpha} + (1-\delta)\bar{k} - (\bar{k} - \epsilon) \\
c_t &= (\bar{k} - \epsilon)^{\alpha} - \delta(\bar{k} - \epsilon), \text{ for } t \geq 1
\end{aligned}$$

   Now let's look at the utility derived from these two strategies:

   - Strategy 1:

$$\begin{aligned}
\mathcal{U}_1 &= \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \beta^t \frac{\left(\bar{k}^{\alpha} - \delta\bar{k}\right)^{1-\sigma} - 1}{1-\sigma} \\
&= \frac{\left(\bar{k}^{\alpha} - \delta\bar{k}\right)^{1-\sigma} - 1}{(1-\sigma)(1-\beta)}
\end{aligned}$$

- Strategy 2:

$$\begin{aligned}
\mathcal{U}_2 &= \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\
&= \frac{\left(\bar{k}^\alpha + (1-\delta)\bar{k} - (\bar{k}-\epsilon)\right)^{1-\sigma} - 1}{1-\sigma} + \beta \sum_{t=0}^{\infty} \beta^t \frac{\left((\bar{k}-\epsilon)^\alpha - \delta(\bar{k}-\epsilon)\right)^{1-\sigma} - 1}{1-\sigma} \\
&= \frac{\left(\bar{k}^\alpha + (1-\delta)\bar{k} - (\bar{k}-\epsilon)\right)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{\left((\bar{k}-\epsilon)^\alpha - \delta(\bar{k}-\epsilon)\right)^{1-\sigma} - 1}{(1-\sigma)(1-\beta)}
\end{aligned}$$

Realise that for $\epsilon = 0$ the two strategies are identical and yield the same utility. Now let's look at the partial derivative of strategy 2 with respect to ϵ evaluated at $\epsilon = 0$:

$$\begin{aligned}
\left.\frac{\partial \mathcal{U}_2}{\partial \epsilon}\right|_{\epsilon=0} &= \left(\bar{k}^\alpha + (1-\delta)\bar{k} - (\bar{k}-\epsilon)\right)^{-\sigma} + \frac{\beta}{1-\beta}\left((\bar{k}-\epsilon)^\alpha - \delta(\bar{k}-\epsilon)\right)^{-\sigma}\left(\alpha(\bar{k}-\epsilon)^{\alpha-1} - \delta\right)(-1) \\
&= \left(\bar{k}^\alpha - \delta\bar{k}\right)^{-\sigma} - \frac{\beta}{1-\beta}\left(\bar{k}^\alpha - \delta\bar{k}\right)^{-\sigma}\left(\alpha\bar{k}^{\alpha-1} - \delta\right) \qquad (15)
\end{aligned}$$

Now realize that at the steady state level of capital, $1 = \beta(1 + f_k(k^{\text{ss}}) - \delta)$ or equivalently, $\frac{1-\beta}{\beta} = \alpha(k^{\text{ss}})^{\alpha-1} - \delta$. However, for $\bar{k} > k^{\text{ss}}$ it follows that $\frac{1-\beta}{\beta} > \alpha(\bar{k})^{\alpha-1} - \delta$. This implies that for $\bar{k} = k^{\text{ss}}$ equation (15) is equal to zero, however for $\bar{k} > k^{\text{ss}}$ equation (15) is positive and therefore strategy 2 is making the consumer strictly better off than strategy 1. We have shown that the consumer is better off, if he keeps a capital level that is marginally below the golden rule capital level given the initial level of capital is the golden rule. In fact we have shown it for any level of capital such that $k^{\text{ss}} < \bar{k} \leq k^{\text{gr}}$.

3 Solution: General Equilibrium: Phase diagram and model dynamics

Consider the neoclassical growth model from exercise 2.

1. *Show the dynamics within the phase diagram and draw the saddle path.*

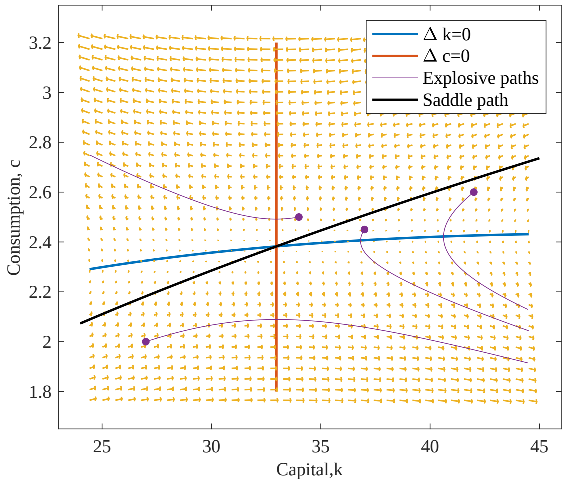
   Solution:

   The resource constraint from (11) indicates the curve, for which capital stays constant, i.e. it depicts the nullcline for capital. Equivalently, the euler equation from (12) defines the level of capital for which consumption is constant, i.e. the nullcline for consumption. These optimality conditions allow us to study the dynamics of the economy when it is not in the steady state, for example after a technology shock. From the resource constraint and the euler equation we can derive the dynamics of consumption and capital, which in turn will give us all remaining variables given an inital level of capital:

$$\begin{aligned}
k_{t+1} &= (1-\delta)k_t + f(k_t, 1) - c_t \\
\frac{k_{t+1}}{k_t} &= \frac{f(k_t,1)}{k_t} - \frac{c_t}{k_t} + 1 - \delta \\
\frac{k_{t+1}}{k_t} &= k_t^{\alpha-1} - \frac{c_t}{k_t} - \delta \qquad (16) \\
u'(c_t) &= \beta(1 + f_k(k_{t+1}, 1) - \delta)u'(c_{t+1}) \\
c_t^{-\gamma} &= \beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta)c_{t+1}^{-\gamma} \\
\frac{c_{t+1}}{c_t} &= \left(\beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta)\right)^{\frac{1}{\gamma}} \qquad (17)
\end{aligned}$$

   Realise that (16) and (17) are both equal to zero if and only if capital and consumption are at their steady state levels. For all other combinations of $\{k_t, c_t\}$ the economy is dynamic and moving along some optimal path. To figure out the dynamics of this economy we can study the two equations and find out how the economy develops. The simplest way to do this, is with the help of a phase diagram shown in figure 2. The little arrows, indicate the direction and speed at which the economy

Figure 2: Phase Diagram



is changing. We can divide the graph into four different areas around the steady state to study the evolvement of the economy, where we refer to the values that lie on the nullcline as k^* and c^*:

- North-east: This means that $k_t > k^*$ and $c_t > c^*$, from (16) and (17) we know that $\frac{k_{t+1}}{k_t} < 0$ and $\frac{c_{t+1}}{c_t} < 0$
- North-west: This means that $k_t < k^*$ and $c_t > c^*$, from (16) and (17) we know that $\frac{k_{t+1}}{k_t} < 0$ and $\frac{c_{t+1}}{c_t} > 0$
- South-east: This means that $k_t > k^*$ and $c_t < c^*$, from (16) and (17) we know that $\frac{k_{t+1}}{k_t} < 0$ and $\frac{c_{t+1}}{c_t} > 0$
- South-west: This means that $k_t < k^*$ and $c_t < c^*$, from (16) and (17) we know that $\frac{k_{t+1}}{k_t} > 0$ and $\frac{c_{t+1}}{c_t} > 0$

Together these observations help us to find out where the saddle path must lie. The saddle path is a (unique) path that shows how the economy converges to the steady state. The saddle path is pinned down by the transversality condition and the equations (16) and (17). All other paths violate the transversality condition and are therefore not optimal. This implies that for a given k_0 consumption c_0 is a jump variable and chosen such that the bundle $\{k_0, c_0\}$ lies on the saddle path.

2. *Suppose that the economy is in the steady state. How do k_{t+1}, c_t, w_t and R_t respond to the following, somewhat model-inconsistent shocks?*

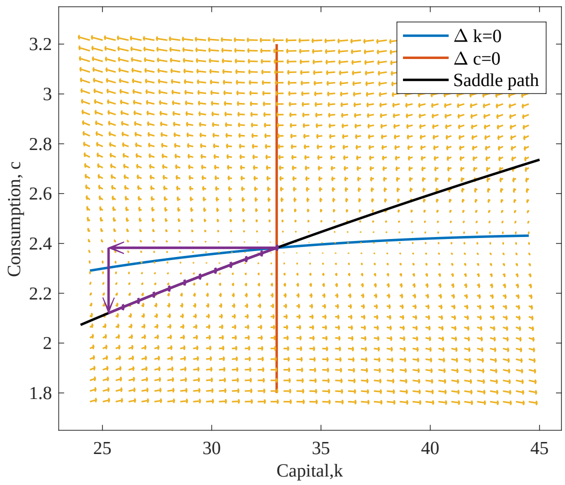
   Draw the adjustment path. Moreover, explain the adjustments from the household's point of view.

   (a) *An earthquake destroys some of the initial (steady state) capital stock.*

   Solution:

   The destruction of capital stock reflects a loss of wealth and additionally it lowers the income because less capital can be rented out to firms. Further, the marginal product of labor decreases, which lowers the households earnings on labor. Although capital can be rented out at a higher price, this does not make up for the losses and the household drops the consumption level. As capital can be rented out a high price, r_t is large, the consumer accumulates capital through saving. Additionally, the high rental rate allows the consumer to spend more on consumption. This implies that both capital and consumption is increasing. As the capital stock is slowly recovering the rental rate is decreasing until eventually the economy is back at the steady state and $c_{t+1} = c_t$, $k_{t+1} = k_t$. In figure 3 and 6 we see the dynamics of the economy after the shock and how the individual variables evolve. Note the jump in consumption immediately after the economy is hit by the shock.

Figure 3: Evolution of the economy after destruction of capital stock



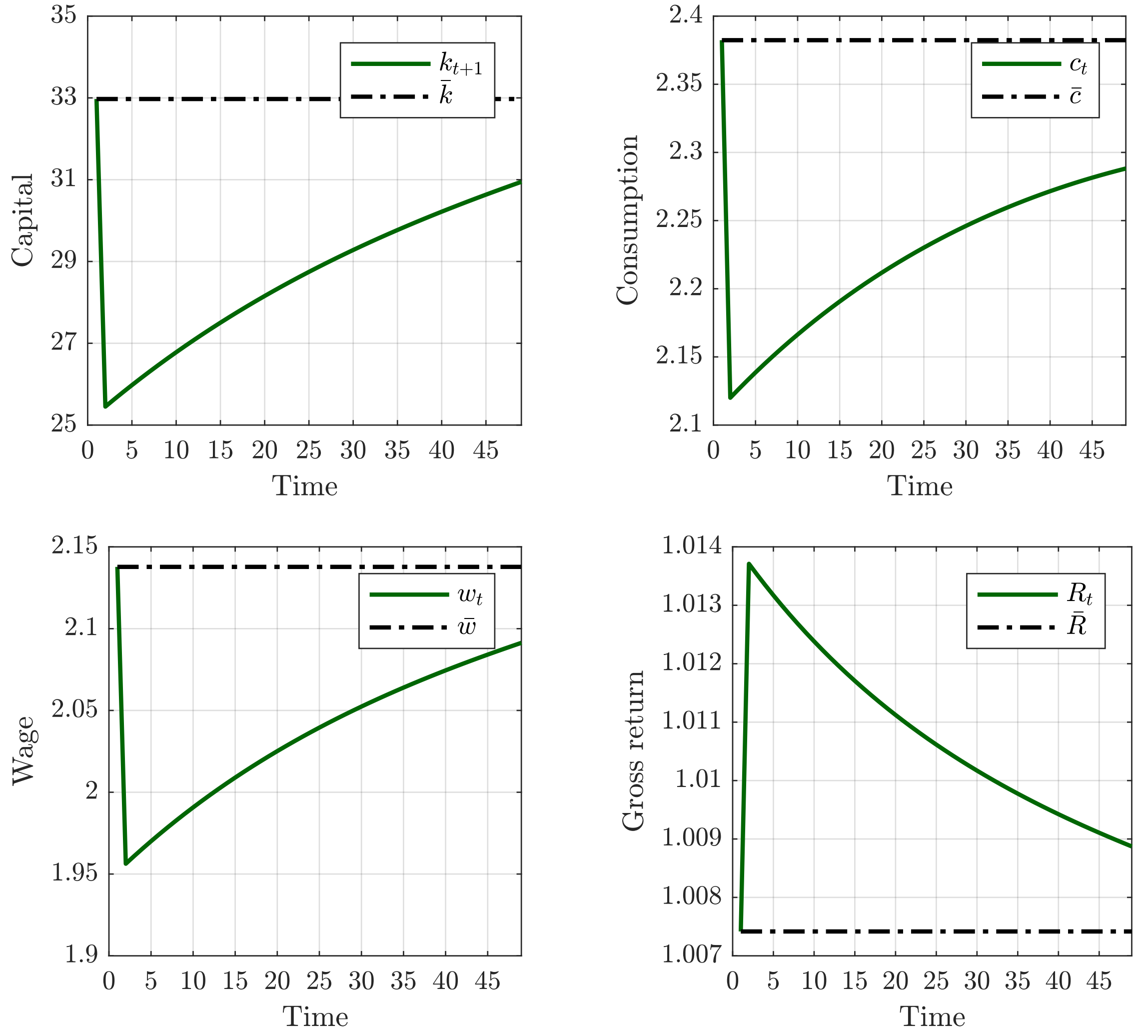


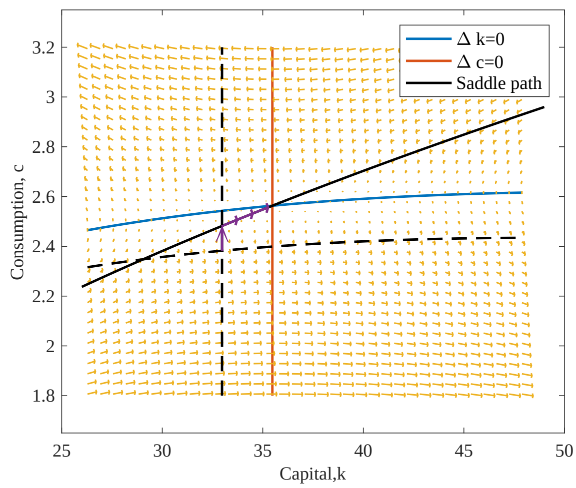
Figure 4: Impulse responses after destruction of capital stock

(b) *There is a onetime, permanent increase in technology a, i.e. the production function changes from $f(k_t, 1)$ to $a \cdot f(k_t, 1)$, with $a > 1$.*

Solution:

The unexpected increase in technology leads to a new steady state in consumption and capital. Because firms, have access to an improved technology in the production process, the steady state capital is larger than the previous level. The increase in the capital steady state, makes labor mor productive which leads to a higher wage. Together, this increases the income of the household, which allows it to spend more on consumption and increases the steady state level of consumption. In figure 5 and 6 we see the dynamics of the economy and the evolution of the different variables. Immediately after the improvement in the technology we see a jump in consumption. As labor income and rental rate go up, the households experience an increase in income which allows it to spend more on consumption. Due to the high return on capital, consumers save some of the additional income and invest in a higher capital stock. As they accumulate capital, the return on capital is slowly decreasing. However, the additional capital makes labor more productive which increases the households labor income. Over time, we see that capital is accumulated until the rental rate has returned to its previous level. In fact the households invests in additional capital until the gross return on capital is equal to the inverse of the discount factor ($\bar{R} = \frac{1}{\beta}$ at which point, the marginal utility of an additional unit of consumption tomorrow is equal to the marginal utility of consumption today.

Figure 5: Evolution of the economy after a permanent technology shock



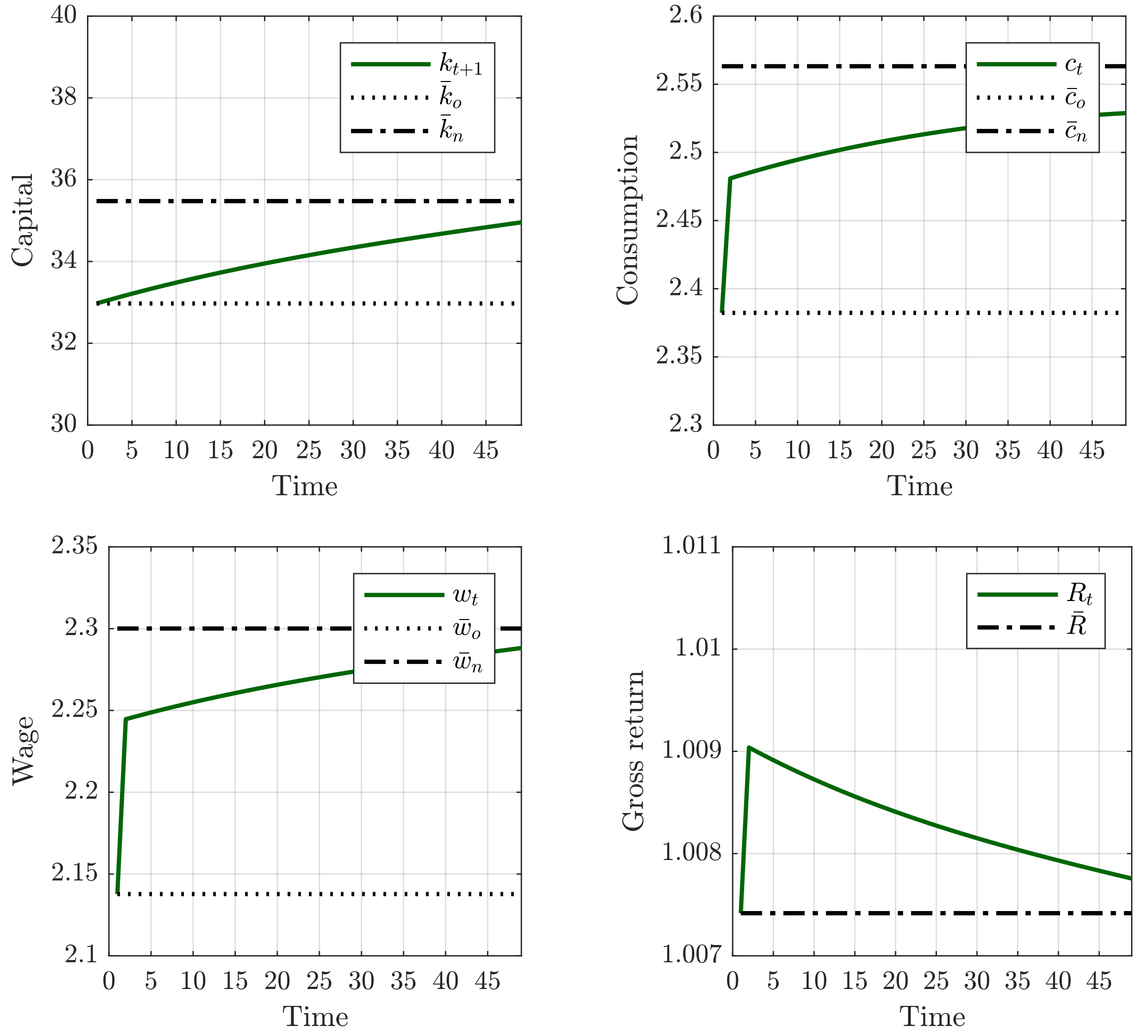


Figure 6: Impulse responses after a permanent technology shock