Problem Set V

Macroeconomics II

Solutions

Armando Näf*

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1 Solution: General equilibrium in an OLG Model

Consider a standard OLG model we studied in class with two cohorts, young and old. Households are infinitely lived with utilities such that $u'(\cdot) > 0$ and $u''(\cdot) < 0$. Firms have access to a constant return to scale technology.

1. *Assume that the population is constant. State the conditions that arise from*

   (a) *utility maximization of households,*

   Solution:

$$k_{t+1} = w_t - c_{1,t} \tag{1}$$

$$c_{2,t} = k_t(1 + r_t - \delta) + \pi_t \tag{2}$$

$$u'(c_{1,t}) = \beta(1 + r_{t+1} - \delta)u'(c_{2,t+1}) \tag{3}$$

   Where, $c_{1,t}$ and $c_{2,t}$ denote consumption at date t of a young and old household, respectively. Equations (1), (2) and (3) are the dynamic budget constraint of a young household, consumption of an old household and the euler equation of a young household.

   (b) *profit maximization of firms, and*

   Solution:

   The firm side is identical to the representative agent model, where the firm simply maximizes its profit function choosing the input factor needed for production: $\max_{K_t,L_t} f(K_t, L_t) - w_t L_t - r_t K_t$. This yields equilibrium factory payments plus profits:

$$w_t = f_L(K_t, L_t) \tag{4}$$

$$r_t = f_K(K_t, L_t) \tag{5}$$

$$\pi_t = f(K_t, L_t) - w_t L_t - r_t K_t \tag{6}$$

   (c) *feasibility.*

   Solution:

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

From the market clearing condition we can pin down L_t and K_t:

$$L_t = 1 \tag{7}$$

$$K_t = k_t \tag{8}$$

Combining these two conditions with the conditions from (1), (2), (4), (5) and (6) we can derive the same Resource constraint as in the representative agent model:

$$k_{t+1} = w_t - c_{1,t} + k_t(1 + r_t - \delta) + \pi_t - c_{2,}$$

$$k_{t+1} = k_t(1-\delta) + f(k_t, 1) - \underbrace{c_t}_{=c_{1,t}+c_{2,t}} \tag{9}$$

2. *Combine these conditions to derive the three core equations that characterize the OLG model without population growth.*

   Solution:

   Combining equations (1), (2), (3), (6), (9) and the market clearing conditions we can derive the three core equations:

$$k_{t+1} \stackrel{(9)}{=} k_t(1-\delta) + f(k_t, 1) - c_{1,t} - c_{2,t}, \tag{10}$$

$$c_{2,t} \stackrel{(2)+(5)+(8)+(7)}{=} k_t(1 + f_k(k_t, 1) - \delta), \tag{11}$$

$$u'(c_{1,t}) \stackrel{(3)+(5)+(8)+(7)}{=} \beta(1 + f_k(k_{t+1}, 1) - \delta)u'(c_{2,t+1}). \tag{12}$$

2 Solution: Law of motion for capital, equilibrium allocation and prices

Consider an OLG model like the one we discussed in 1 without population growth. Assume logarithmic preferences $u(c_t) = \ln(c_t)$.

1. *Show that the equilibrium law of motion for capital is given by $k_{t+1} = \frac{\beta}{1+\beta} f_L(k_t, 1)$.*

   Solution: The maximisation problem of the household is given by

$$\max_{c_{1,t}, c_{2,t+1}, a_{t+1}} u(c_{1,t}) + \beta u(c_{2,t+1}),$$

$$\text{s.t. } c_{1,t} + a_{t+1} = w_t,$$

$$c_{2,t+1} = R_{t+1}a_{t+1} + \pi_t,$$

   where we can replace $c_{1,t} = w_t - a_{t+1}$ and $c_{2,t+1} = R_{t+1}a_{t+1}$ (note that $\pi_t = 0$) to get an unconstrained optimization problem with a_{t+1} as the only choice variable

$$\max_{a_{t+1}} u(w_t - a_{t+1}) + \beta u(R_{t+1}a_{t+1}).$$

   The first order condition of this problem is then given by

$$u'(w_t - a_{t+1})(-1) + \beta u'(R_{t+1}a_{t+1})R_{t+1} \stackrel{!}{=} 0.$$

For logarithmic preferences we then find

$$\begin{aligned}\frac{1}{w_t - a_{t+1}} &= \beta \frac{R_{t+1}}{R_{t+1}a_{t+1}} \\ a_{t+1} &= \beta(w_t - a_{t+1}) \\ a_{t+1} &= \frac{\beta}{1+\beta} w_t \\ a_{t+1} &= \frac{\beta}{1+\beta} f_L(k_t, 1) \\ k_{t+1} &= \frac{\beta}{1+\beta} f_L(k_t, 1) \qquad (13)\end{aligned}$$

Where we replaced w_t with $f_L(k_t, 1)$ from the firm's optimization problem and a_{t+1} with k_{t+1} from the market clearing condition since k_t is the only asset.

2. *Characterize the equilibrium path of k_t graphically assuming the Cobb-Douglas production function $f(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$.*

   Solution:

   From (13) we know that the dynamics of k_{t+1} are

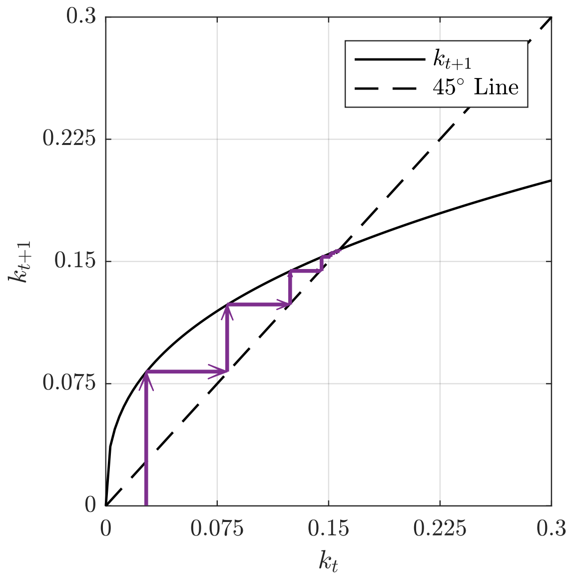
$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

   with the steady state of k given by

$$k^{\text{ss}} = \left(\frac{(1-\alpha)\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$$

   Realise that this is different from the steady state capital with a representative agent economy. Given an inital level of capital k_0, the dynamics of capital are shown in figure 1

Figure 1: Dynamics of k_{t+1}



3. *Characterize the equilibrium allocation and prices.*

   Solution:

Given k_0

$$
\begin{aligned}
w_t &= f_L(k_t, 1) \\
r_t &= f_K(k_t, 1) \\
k_{t+1} &= \frac{\beta}{1+\beta} f_L(k_t, 1) \\
c_{1,t} &= w_t - k_{t+1} = \frac{1}{1+\beta} f_L(k_t, 1) \\
c_{2,t} &= k_t(1 + f_K(k_t, 1) - \delta)
\end{aligned}
$$

3 Solution: Individual and aggregate savings

Suppose that young households save a constant fraction of their labor income for retirement. Assume that there is a constant population growth ν, such that $N_{1,t} = \nu N_{2,t}$, where $N_{i,t}$ is the size of cohort i at time t.

1. *Define savings of young and old households as well as aggregate savings.*

   Solution:

   Savings are equal to income minus consumption. This implies the following:

$$
\begin{aligned}
\text{young: } \quad & a_{t+1} = w_t - c_{1,t} \\
\text{old: } \quad & a_t r_t - c_{2,t} = a_t(R_t - 1) - c_{2,t} \\
& = a_t(R_t - 1) - a_t R_t \\
& = -a_t
\end{aligned}
$$

   where the third line follows from the fact that old agents will consume all the assets they brought into the period.

   Aggregate savings are then given by the savings from the young agents plus the savings from the old agents:

$$
\begin{aligned}
N_{1,t} a_{t+1} + N_{2,t}(-a_t) &= \nu N_{2,t} a_{t+1} + N_{2,t}(-a_t) \\
&= N_{2,t}(\nu \phi w_t - \phi w_{t-1}) \qquad (14)
\end{aligned}
$$

   where the last line follows from the assumption that young households save a constant fraction ϕ from their wage. In the case with logarithmic utility from question 2 we showed that $\phi = \frac{\beta}{1+\beta}$.

   Note: with population growth $a_{t+1} \neq k_{t+1}$. Instead we have $a_{t+1} = \nu k_{t+1}$ since k_{t+1} is the capital per worker and todays savings a_{t+1} per worker will have to be shared by a larger working group. In other words k_{t+1} is capital per worker at time $t+1$ and a_{t+1} are savings per worker at time t.

2. *How do aggregate savings evolve over time if the population size is either constant or growing and wages are either constant or growing?*

   Solution:

   Suppose that $N_{1,t} = (1+n_t)N_{2,t}$ and $w_{t+1} = (1+g_{t+1})w_t$ such that the growth rate of the population is n_t and the growth rate of wages is g_t (where we use the subscript to allow for a more general discussion rather than looking only at constant population growth). From (14) we find

$$\begin{aligned}
N_{1,t}a_{t+1} + N_{2,t}(-a_t) &= N_{2,t}((1+n_t)\phi w_t - \phi w_{t-1}) \\
&= N_{2,t}((1+n_t)(1+g_t)\phi w_{t-1} - \phi w_{t-1}) \\
&= \underbrace{N_{2,t}\phi w_{t-1}}_{=N_{2,t}a_t}((1+n_t)(1+g_t) - 1) \\
\frac{N_{1,t}a_{t+1} + N_{2,t}(-a_t)}{N_{2,t}a_t} &= (1+n_t)(1+g_t) - 1 \\
\frac{N_{1,t}(1+n_{t+1})k_{t+1} + N_{2,t}(1+n_t)(-k_t)}{N_{2,t}(1+n_t)k_t} &= (1+n_t)(1+g_t) - 1 \\
\frac{N_{1,t}(1+n_{t+1})\frac{K_{t+1}}{N_{1,t+1}} + N_{2,t}(1+n_t)\left(-\frac{K_t}{N_{1,t}}\right)}{N_{2,t}(1+n_t)\frac{K_t}{N_{1,t}}} &= (1+n_t)(1+g_t) - 1 \\
\frac{N_{1,t}(1+n_{t+1})\frac{K_{t+1}}{N_{1,t}(1+n_{t+1})} + N_{2,t}(1+n_t)\left(-\frac{K_t}{N_{2,t}(1+n_t)}\right)}{N_{2,t}(1+n_t)\frac{K_t}{N_{2,t}(1+n_t)}} &= (1+n_t)(1+g_t) - 1 \\
\frac{K_{t+1} - K_t}{K_t} &= (1+n_t)(1+g_t) - 1 \qquad (15)
\end{aligned}$$

where we used the information that $a_{t+1} = (1+n_{t+1})k_{t+1} = (1+n_{t+1})\frac{K_{t+1}}{N_{1,t+1}}$, and $N_{1,t+1} = (1+n_{t+1})N_{1,t}$ hence $N_{1,t}a_{t+1} = (1+n_{t+1})N_{1,t}\frac{K_{t+1}}{N_{1,t+1}} = K_{t+1}$ It follows that growth rate of aggregate savings (or capital) is following the path determined by (15).

4 Solution: Government intervention

Consider an infinite-horizon OLG model without population growth in which capital does not contribute to production (i.e., $f(K_t, L_t) = L_t$ or $f(k_t, 1) = 1$), but may be stored from one period to the next with depreciation rate δ.

1. *Show that the steady state of this economy is necessarily dynamically inefficient.*

   Solution:

   Taking the derivative of the steady state resource constraint, $c = f(k,1) - \delta k = 1 - \delta k$, with respect to k, we get

   $$\frac{\partial c}{\partial k} = -\delta < 0.$$

   Hence, the economy is dynamically inefficient. In other words, because the storage technolog has a return that is smaller than 1, each stored unit is deteriorating over one period which is inefficient from a social planner perspective.

Suppose that there is a government transferring b from young to old households in each period.

2. *Which condition needs to hold for a marginal increase in b leading to a Pareto improvement? Does it hold for the economy in this exercise?*

   Solution:

   For $b > 0$, the initial old generation clearly benefits as it receives a higher transfer without having paid more into the social security system. The initial young and all future generations have to pay more into the system when they are young and they get a larger social security transfer when they are old. For them to be better off, the following condition needs to hold:

$$\frac{\partial \left[u(\overbrace{w_t - a^*_{t+1} - b}^{=c_{1,t}}) + \beta u(\overbrace{a^*_{t+1} R_{t+1} + b}^{=c_{2,t+1}}) \right]}{\partial b} = -u'(c_{1,t}) + \beta u'(c_{2,t+1}) \geq 0 \tag{16}$$

Note that the optimal savings decision a^*_{t+1} in general depends on b. However, by the envelope theorem, we do not need to take the terms $\frac{\partial a^*_{t+1}}{\partial b}$ into account when computing the marginal effect of a change in b on a generation's utility.

To check if condition (16) holds in the given economy, we use the Euler equation:

$$\begin{aligned} u'(c_{1,t}) &= \beta R_{t+1} u'(c_{2,t+1}) \\ &= \beta(1 - \delta + f_K(k_{t+1}, 1))u'(c_{2,t+1}) \\ &= \beta(1 - \delta)u'(c_{2,t+1}) \\ \beta(1 - \delta)u'(c_{2,t+1}) - u'(c_{1,t}) &= 0 \\ \beta u'(c_{2,t+1} - u'(c_{1,t}) &= \beta\delta u'(c_{2,t+1}) > 0 \end{aligned}$$

Thus, condition (16) holds and a marginal increase in b would benefit all current and future generations as long as the euler equation holds.

3. *What amount of social security transfer b is optimal?*

Solution:

In subquestion 2, we showed that an increase in b benefits all generations as long as the Euler equation holds. Thus, as long as savings are positive, it is optimal to increase the transfer b. The optimal transfer is such that households do not save any more. This is also evident from the resource constraint of this economy, which shows that any positive k destroys resources without adding anything to production. Stated differently, any $k > 0$ is dynamically inefficient and the optimal transfer should be chosen such that $k = 0$.