Problem Set VI

Macroeconomics II

Solutions

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1 Solution: Incomplete markets

There is a representative household living for three periods ($t = 0, 1, 2$). There is only one asset available for saving. The household faces risk regarding future wages and interest rates (i.e., w_1, R_1, w_2 and R_2 are unknown). Suppose that there are two possible states of nature in periods $t = 1$ and $t = 2$, state h and state l, which occur with probabilities π_h and $\pi_l = 1 - \pi_h$, respectively.

1. *Write down the maximization problem of the household. What are the choice variables in this problem?*

   Solution:

   The household chooses consumption and savings to maximize the life time utility, subject to the dynamic budget constraints:

$$
\begin{aligned}
\max_{\{c_0,c_{1,i},c_{2,i,j},a_1,a_{2,i}\}_{i,j\in h,l)}} \quad & u(c_0) + \mathbb{E}_0\left[\beta u(c_{1,i}) + \beta^2 u(c_{2,i,j})\right] \\
\text{s.t.} \quad & a_1 = a_0 R_0 + w_0 - c_0 \\
& a_{2,i} = a_1 R_{1,i} + w_{1,i} - c_{1,i}, \text{for } i \in \{l, h\} \\
& 0 = a_{2,i} R_{2,i,j} + w_{2,i,j} - c_{2,i,j}, \text{for } i, j \in \{l, h\}
\end{aligned}
$$

   The household chooses consumption and savings for time period t and all possible states that can occur.

2. *Solve the maximization problem and derive the three Euler equations.*

   Solution:

   Can replace consumption to reduce the number of choice variables

$$
\max_{\{a_1,a_{2,i}\}_{i=(h,l)}} u(a_0R_0 + w_0 - a_1) + \mathbb{E}_0\left[\beta u(a_1R_{1,i} + w_{1,i} - a_{2,i}) + \beta^2 u(a_{2,i}R_{2,i,j} + w_{2,i,j})\right]
$$

   We then replace the expectations operator by the probabilities that the specific states occur:

*I am sure there are many typos in the script. If you find any please send me an email to armando.naef@vwi.unibe.ch

$$
\begin{aligned}
\max_{\{a_1,a_{2,i}\}_{i\in(h,l)}} \quad & u(a_0R_0 + w_0 - a_1) \\
& + \beta\left(\pi_l u(a_1R_{1,l} + w_{1,l} - a_{2,l}) + \pi_h u(a_1R_{1,h} + w_{1,h} - a_{2,h})\right) \\
& + \beta^2\big(\pi_l^2 u(a_{2,l}R_{2,l,l} + w_{2,l,l}) + \pi_l\pi_h u(a_{2,l}R_{2,l,h} + w_{2,l,h}) \\
& \quad + \pi_h\pi_l u(a_{2,h}R_{2,h,l} + w_{2,h,l}) + \pi_h^2 u(a_{2,h}R_{2,h,h} + w_{2,h,h})\big)
\end{aligned}
$$

The first order conditions are given by

$$
\begin{aligned}
a_1: \quad & u'(a_0R_0 + w_0 - a_1)(-1) + \beta\left(\pi_l u'(a_1R_{1,l} + w_{1,l} - a_{2,l})R_{1,l} + \pi_h u'(a_1R_{1,h} + w_{1,h} - a_{2,h}R_{1,h})\right) \overset{!}{=} 0, \\
a_{2,l}: \quad & \beta(\pi_l u'(a_1R_{1,l} + w_{1,l} - a_{2,l})(-1) + \beta^2\left(\pi_l^2 u'(a_{2,l}R_{2,l,l} + w_{2,l,l})R_{2,l,l}\right. \\
& \left. + \pi_l\pi_h u'(a_{2,l}R_{2,l,h} + w_{2,l,h})R_{2,l,h}\right) \overset{!}{=} 0, \\
a_{2,h}: \quad & \beta(\pi_h u'(a_1R_{1,h} + w_{1,h} - a_{2,h})(-1) + \beta^2\left(\pi_h\pi_l u'(a_{2,h}R_{2,h,l} + w_{2,h,l})R_{2,h,l}\right. \\
& \left. + \pi_h^2 u'(a_{2,h}R_{2,h,h} + w_{2,h,h})R_{2,h,h}\right) \overset{!}{=} 0.
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
a_1: \quad & u'(c_0) = \beta\left(\pi_l u'(c_{1,l})R_{1,l} + \pi_h u'(c_{1,h})R_{1,h})\right), \qquad (1) \\
a_{2,l}: \quad & u'(c_{1,l}) = \beta\left(\pi_l u'(c_{2,l,l})R_{2,l,l} + \pi_h u'(c_{2,l,h})R_{2,l,h}\right), \\
a_{2,h}: \quad & u'(c_{1,h}) = \beta\left(\pi_l u'(c_{2,h,l})R_{2,h,l} + \pi_h u'(c_{2,h,h})R_{2,h,h}\right)
\end{aligned}
$$

Suppose now additionally that $\beta R_t = 1$ for all t, there are no assets to start with ($a_0 = 0$), and utility is quadratic, i.e.,

$$
u(c_t) = \phi c_t - \frac{1}{2}c_t^2,
$$

in which ϕ is large relative to c_t. Wages in period $t = 1$ and $t = 2$ can take the values $w_{1,i}$ and $w_{2,ij}$, respectively, where i indexes the state in period $t = 1$ and j indexes the state in period $t = 2$, with $i, j \in \{h, l\}$.

3. *Show that the quadratic utility function features decreasing, linear marginal utility.*

   Solution:

$$
\begin{aligned}
u'(c_t) &= \phi - c_t < 0, \\
u''(c_t) &= -1
\end{aligned}
$$

4. *Derive the intertemporal budget constraint(s). How many of them are there?*

   Solution:

   Let's recollect the dynamic budget constraints and use the fact that the interest rate is constant and given by $R = \beta^{-1}$

$$
\begin{aligned}
\text{DBC}_0 \quad & a_1 = a_0R + w_0 - c_0, \\
\text{DBC}_{1,i} \quad & a_{2,i} = a_1R + w_{1,i} - c_{1,i}, \text{for } i \in \{l, h\}, \\
\text{DBC}_{1,i,j} \quad & 0 = a_{2,i}R + w_{2,i,j} - c_{2,i,j}, \text{for } i, j \in \{l, h\}.
\end{aligned}
$$

It follows

$$
\begin{aligned}
\text{DBC}_0 \quad & a_1 = \underbrace{a_0}_{=0} R + w_0 - c_0, \\
\text{DBC}_{1,i} \quad & a_{2,i} = (\underbrace{w_0 - c_0}_{=a_1})R + w_{1,i} - c_{1,i}, \text{for } i \in \{l,h\}, \\
\text{DBC}_{1,i,j} \quad & 0 = (\underbrace{(w_0 - c_0)R + w_{1,i} - c_{1,i}}_{a_{2,i}})R + w_{2,i,j} - c_{2,i,j}, \text{for } i,j \in \{l,h\}, \\
\text{IBC}_{i,j} \quad & c_0 + \frac{c_{1,i}}{R} + \frac{c_{2,i,j}}{R^2} = w_0 + \frac{c_{w,i}}{R} + \frac{w_{2,i,j}}{R^2}, \text{for } i,j \in \{l,h\}.
\end{aligned}
$$

Thus we have four different intertemporal budget constraints for all possible combinations of states.

5. *Use the Euler equations to show that* $\mathbb{E}_0 [c_1 - c_0] = 0$.

   Solution:

   In (1) we derived the euler equation for period $t = 0$

$$
\begin{aligned}
u'(c_0) &= \beta \left(\pi_l u'(c_{1,l}) R_{1,l} + \pi_h u'(c_{1,h}) R_{1,h}\right) \\
&= \beta \, \mathbb{E}\left[u'(c_1) R_1\right] \\
(\phi - c_0) &= \beta \, \mathbb{E}\left[(\phi - c_1) R\right] \\
\mathbb{E}[c_1 - c_0] &= 0.
\end{aligned}
$$

   We made use of the specific functional form of $u(\cdot)$ to find the marginal utilities and used the information from the exercise that $\beta R = 1$ to derive the last line. In other words we find that $c_0 = \mathbb{E}[c_1]$ which implies that the consumption path follows a random walk.

6. *Using the intertemporal budget constraints as of time $t = 0$ and $t = 1$, derive the optimal consumption c_0^*, $c_{1,h}^*$ and $c_{1,l}^*$.*

   Solution:

   At time $t = 0$ the IBC is given by the expectation about future outcomes, given today's information set it follows that

$$
c_0 + \frac{\mathbb{E}_0[c_1]}{R} + \frac{\mathbb{E}_0[c_2]}{R^2} = w_0 + \frac{\mathbb{E}_0[w_1]}{R} + \frac{\mathbb{E}_0[w_2]}{R^2}.
$$

   From the euler equation at period $t = 0$ we know that $c_0 = \mathbb{E}_0[c_1]$ and from the euler equation at period $t = 1$ we know that $c_{1,i} = E_1[c_2]$. By the law of iterated expectations it follows that $c_0 = \mathbb{E}_0\left[E_1[c_2]\right] = E_0[c_2]$ We can then use these results to replace them in the IBC at period $t = 0$ and solve for consumption c_0:

$$
\begin{aligned}
c_0^* + \frac{\mathbb{E}_0[c_1]}{R} + \frac{\mathbb{E}_0[c_2]}{R^2} &= w_0 + \frac{\mathbb{E}_0[w_1]}{R} + \frac{\mathbb{E}_0[w_2]}{R^2} \\
c_0^* + \frac{c_0}{R} + \frac{c_0}{R^2} &= w_0 + \frac{\mathbb{E}_0[w_1]}{R} + \frac{\mathbb{E}_0[w_2]}{R^2} \\
c_0^* &= \frac{w_0 + \frac{\mathbb{E}_0[w_1]}{R} + \frac{\mathbb{E}_0[w_2]}{R^2}}{1 + \frac{1}{R} + \frac{1}{R^2}} \\
c_0^* &= \frac{w_0 + \beta \, \mathbb{E}_0[w_1] + \beta^2 \, \mathbb{E}_0[w_2]}{1 + \beta + \beta^2} \qquad (2)
\end{aligned}
$$

Which implies that $a_1^* = w_0 - c_1^*$. Using the result from (2) and the IBC at period $t = 1$, we can then solve for $c_{1,i}^*$:

$$
\begin{aligned}
c_0^* + \frac{c_{1,i}^*}{R} + \frac{\mathbb{E}_1[c_2]}{R^2} &= w_0 + \frac{w_{1,i}}{R} + \frac{\mathbb{E}_1[w_2]}{R^2} \\
c_0^* + \frac{c_{1,i}^*}{R} + \frac{c_{1,i}^*}{R^2} &= w_0 + \frac{w_{1,i}}{R} + \frac{\mathbb{E}_1[w_2]}{R^2} \\
c_{1,i}^* &= \frac{a_1^* + \frac{w_{1,i}}{R} + \frac{E_1[w_2]}{R^2}}{\frac{1}{R} + \frac{1}{R^2}} \\
c_{1,i}^* &= \frac{a_1^* + \beta w_{1,i} + \beta^2 \mathbb{E}_1[w_2]}{\beta + \beta^2}, \text{ for } i \in \{l, h\} \qquad (3)
\end{aligned}
$$

Realise that $c_{1,l}^* \neq c_{1,h}^*$ but depends on the realization of the wage at time $t = 1$ and the adjusted expectation for the wage at time $t = 2$ conditional on what state the economy is in at period $t = 1$.

7. *What is the sign and the magnitude of $c_{1,i}^* - c_0^*, i = h, l$?*

Solution:

From (3) we can solve for the change in consumption from period $t = 0$ to $t = 1$:

$$
\begin{aligned}
c_{1,i}^* &= \frac{\frac{a_1^*}{\beta} + w_{1,i} + \beta \mathbb{E}_1[w_2]}{1+\beta}, \text{ for } i \in \{l, h\} \\
c_{1,i}^* - c_0^* &= \frac{\frac{a_1^*}{\beta} + w_{1,i} + \beta \mathbb{E}_1[w_2]}{1+\beta} - c_0^*, \text{ for } i \in \{l, h\} \\
(1+\beta)(c_{1,i}^* - c_0^*) &= \frac{a_1^*}{\beta} + w_{1,i} + \beta \mathbb{E}_1[w_2] - (1+\beta)c_0^*, \text{ for } i \in \{l, h\} \\
&= \frac{w_0 - c_1^*}{\beta} + w_{1,i} + \beta \mathbb{E}_1[w_2] - (1+\beta)\left(\frac{w_0 + \beta \mathbb{E}_0[w_1] + \beta^2 \mathbb{E}_0[w_2]}{1+\beta+\beta^2}\right), \text{ for } i \in \{l, h\} \\
&= \frac{w_0}{\beta} + w_{1,i} + \beta \mathbb{E}_1[w_2] - \frac{1}{\beta}(1+\beta+\beta^2)\left(\frac{w_0 + \beta \mathbb{E}_0[w_1] + \beta^2 \mathbb{E}_0[w_2]}{1+\beta+\beta^2}\right), \text{ for } i \in \{l, h\} \\
&= \frac{w_0}{\beta} + w_{1,i} + \beta \mathbb{E}_1[w_2] - \frac{1}{\beta}\left(w_0 + \beta \mathbb{E}_0[w_1] + \beta^2 \mathbb{E}_0[w_2]\right), \text{ for } i \in \{l, h\} \\
c_{1,i}^* - c_0^* &= \frac{w_{1,i} + \beta \mathbb{E}_1[w_2]}{1+\beta} - \frac{(\mathbb{E}_0[w_1] + \beta \mathbb{E}_0[w_2])}{1+\beta}, \text{ for } i \in \{l, h\} \\
c_{1,i}^* - c_0^* &= \frac{1}{1+\beta}\left[\left(\underbrace{w_{1,i} - \mathbb{E}_0[w_1]}_{\text{Unexpected change in income}}\right) + \beta\left(\underbrace{\mathbb{E}_1[w_2] - \mathbb{E}_0[w_2]}_{\text{Change in expectations about future income}}\right)\right], \text{ for } i \in \{l, h\} \qquad (4)
\end{aligned}
$$

Expectations about the change in consumption is zero, i.e. $\mathbb{E}[c_1^* - c_0^*] = 0$. However, we see from (4) that $c_{1,i}^* - c_0^*$ is positive if the realization of $w_{1,i}$ (i) is higher than expected and (ii) reveals new (positive) information about w_2. In other words if state h occurs then consumption in period $t = 1$ is higher than expected because wages are higher, and may further increase c_1 if there is some persistence in the states, which implies that $\mathbb{E}_1[w_2] - \mathbb{E}_0[w_2]$ is positive.

2 Solution: Complete markets

Consider a two-period model with two states of nature in the second period, h and l, occurring with probabilities π_h and $\pi_l = 1 - \pi_h$ and in which wages amount to $w_1(h)$ and $w_1(l)$, respectively. There are two assets, $a_1^{(1)}$ and $a_1^{(2)}$. In period $t = 0$ the household starts with wage w_0 and initial assets a_0.

1. *Suppose that the return vectors of the assets are given by [1 r] and [s 1], respectively, across the two states h and l. If $r = s = 0$, these assets are Arrow securities. However, Arrow securities are not*

required for market completeness. Are markets complete if $r = s = 1$? What if $r = 1$ and $s = 0.5$? What if $r = \frac{1}{s}$?

Solution:

For markets to be complete, it must be that the households can fully hedge all risk with the available assets. This is the case if the following two conditions hold

(a) There must be an asset for each possible state

(b) The assets must be linearly independent

The first one is very intuitive, if there are less assets available than possible states, there must be at least one outcome for which the household cannot have an insurance. This implies that there is uncertainty about the future wealth/income and markets are therefore incomplete. The second condition means that none of the assets can be *copied* by the others. In other words, if there are s possible states and s available assets, then if markets are complete, neither of the assets can be copied by a linear combination of the remaining assets. The second condition, implies that $det([R^{(1)}\ R^{(2)}]) \neq 0$, where $R^{(i)}$ is the return vector of asset i.

The simplest example of this, are arrow securities. Each arrow security has a return of one in one specific state. Clearly the two conditions than hold, because there is one asset for each state and none of the assets can be copied by the others since every asset pays a return in only one state. In that case the matrix $[R^{(1)}\ R^{(2)}]$ is just a diagonal matrix with each diagonal element equal to one.

- What if $r = s = 1$? Clearly we have an asset for every state, since there are two states and two assets. But what about linear independence?

$$[R^{(1)}\ R^{(2)}] = \begin{bmatrix} 1 & s \\ r & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

These assets are not linearly independent, because they are simple copies of each other. The determinant of this matrix is equal to zero and therefore markets are incomplete.

- What if $r = 1, s = 0.5$? Clearly we have an asset for every state, since there are two states and two assets. But what about linear independence?

$$[R^{(1)}\ R^{(2)}] = \begin{bmatrix} 1 & s \\ r & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 1 & 1 \end{bmatrix}$$

This determinant is given by $det([R^{(1)}\ R^{(2)}]) = 0.5 \neq 0$. It follows that the two assets are linearly independent and markets are complete. In fact we could use these two assets to replicate the Arrow securities, by buying two of the first assets and selling two of the second, which would yield a return of 1 in state 1 and 0 in state 2, further if we sell one of the first asset and buy two of the second we have another portfolio that yields 0 in state 1 and 1 in state 2.

- What if $r = \frac{1}{s}$? Clearly we have an asset for every state, since there are two states and two assets. But what about linear independence?

$$[R^{(1)}\ R^{(2)}] = \begin{bmatrix} 1 & s \\ r & 1 \end{bmatrix} = \begin{bmatrix} 1 & s \\ \frac{1}{s} & 1 \end{bmatrix}$$

These assets are not linearly independent, because the second asset is simply the first asset scaled by s. The determinant of this matrix is equal to zero and therefore markets are incomplete.

2. *Write down the dynamic budget constraints given return vectors* $[R_1^{(1)}(h)\ R_1^{(1)}(l)]$ *and* $[R_1^{(2)}(h)\ R_1^{(2)}(l)]$.

Solution:

$$\begin{aligned} t=0: &\quad a_1^{(1)} + a_1^{(2)} = w_0 + a_0R_0 - c_0 \\ t=1: &\quad c_1(s) = w_1(s) + a_1^{(1)}R_1^{(1)}(s) + a_1^{(2)}R_1^{(2)}(s), \text{ for } s = l, h \end{aligned}$$

3. *Suppose that the return vectors are* $[R_1^{(1)}(h)\ 0]$ *and* $[0\ R_1^{(2)}(l)]$. *Solve the maximization problem of the household and derive the Euler equations.*

Solution:

$$\begin{aligned} \max_{\left\{c_0, c_1(s), a_1^{(1)}, a_2^{(2)}\right\}_{s\in h,l)}} &\quad u(c_0) + \mathbb{E}_0\left[\beta u(c_1(s))\right] \\ \text{s.t. } &\quad a_1^{(1)} + a_1^{(2)} = w_0 + a_0R_0 - c_0 \\ &\quad c_1(s) = w_1(s) + a_1^{(1)}R_1^{(1)}(s) + a_1^{(2)}R_1^{(2)}(s), \text{ for } s = l, h \end{aligned}$$

Equivalently we can substitute for c_0 and $c_1(s)$ to get an unconstrained maximization problem

$$\begin{aligned} \max_{a_1^{(1)}, a_2^{(2)}} \quad u\left(w_0 + a_0R_0 - a_1^{(1)} - a_1^{(2)}\right) + \beta \Big[\pi_l u\left(w_1(l) + a_1^{(1)}R_1^{(1)}(l) + a_1^{(2)}R_1^{(2)}(l)\right) \\ + \pi_h u\left(w_1(h) + a_1^{(1)}R_1^{(1)}(h) + a_1^{(2)}R_1^{(2)}(h)\right)\Big] \end{aligned}$$

which is equal to

$$\begin{aligned} \max_{a_1^{(1)}, a_2^{(2)}} \quad u\left(w_0 + a_0R_0 - a_1^{(1)} - a_1^{(2)}\right) + \beta \Big[\pi_l u\left(w_1(l) + a_1^{(2)}R_1^{(2)}(l)\right) \\ + \pi_h u\left(w_1(h) + a_1^{(1)}R_1^{(1)}(h)\right)\Big] \end{aligned}$$

because $a_1^{(1)}$ returns 0 in state l and $a_1^{(2)}$ returns 0 in state h. The first order conditions of this problem are given by

$$\begin{aligned} a_1^{(1)}: &\quad u'\left(w_0 + a_0R_0 - a_1^{(1)} - a_1^{(2)}\right)(-1) + \beta\left[\pi_h u'\left(w_1(h) + a_1^{(1)}R_1^{(1)}(h)\right)R_1^{(1)}(h)\right] \overset{!}{=} 0 \\ a_1^{(2)}: &\quad u'\left(w_0 + a_0R_0 - a_1^{(1)} - a_1^{(2)}\right)(-1) + \beta\left[\pi_l u'\left(w_1(l) + a_1^{(2)}R_1^{(2)}(l)\right)R_1^{(2)}(l)\right] \overset{!}{=} 0 \end{aligned}$$

These two conditions yield the two euler equations for the household:

$$\begin{aligned} u'(c_0) &= \beta\left[\pi_h u'(c_1(h))\, R_1^{(1)}(h)\right] \\ u'(c_0) &= \beta\left[\pi_l u'(c_1(l))\, R_1^{(2)}(l)\right] \end{aligned}$$